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AMERICAN MATHEMATICS COMPETITIONS

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THE AKAMAI FOUNDATION

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# AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

# SOLUTIONS PAMPHLET

## Tuesday, April 9, 2002

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

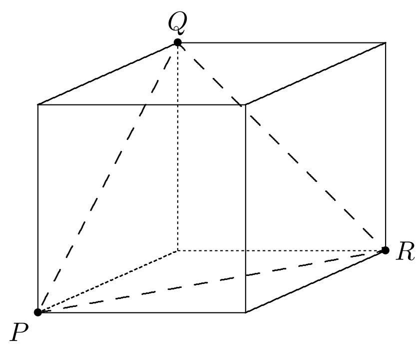
We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

*Correspondence about the problems and solutions should be addressed to:*
**David Hankin, AIME Chair**
**Hunter College High School, Dept. of Mathematics, 71 East 94th St., New York, NY 10128 USA**

*Order prior year Exams, Solutions Pamphlets or Problem Books from:*
**Titu Andreescu, MAA AMC Director**
**University of Nebraska–Lincoln, P.O. Box 81606, Lincoln, NE 68501-1606 USA**

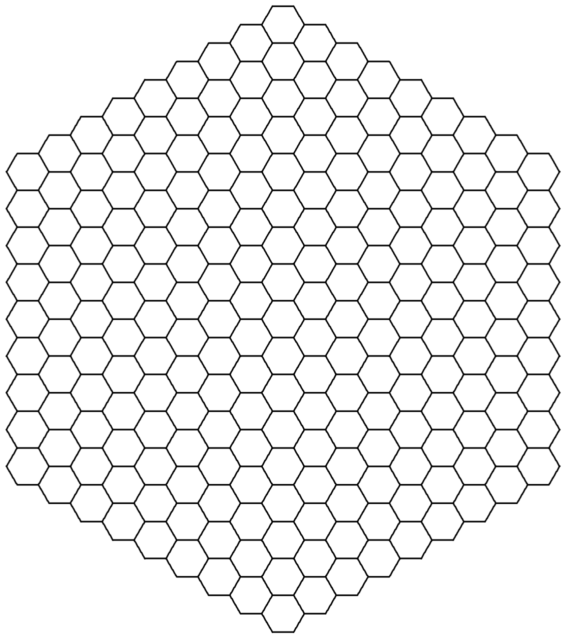
1. (Answer: 009)
   Let the hundreds, tens, and units digits of $x$ be $h$, $t$, and $u$, respectively. Then $x = 100h + 10t + u$, $y = 100u + 10t + h$, and $z = |99(h - u)| = 99|h - u|$. Since $h$ and $u$ are between 1 and 9, inclusive, $|h - u|$ must be between 0 and 8, inclusive. Thus there are 9 possible values for $z$.

2. (Answer: 294)
   Notice that $PQR$ is an equilateral triangle, because $PQ = QR = RP = 7\sqrt{2}$. This implies that each edge of the cube is 7 units long. Hence the surface area of the cube is $6(7^2) = 294$.




3. (Answer: 111)
   Note that $6 = \log_6 a + \log_6 b + \log_6 c = \log_6 abc$. Then $6^6 = abc = b^3$, so $b = 6^2$ and $ac = 6^4$. Because $b \neq a$ and $b - a$ is the square of an integer, the only possibilities for $a$ are 11, 20, 27, 32, and 35. Of these, only 27 is a divisor of $6^4$. Thus $a + b + c = 27 + 36 + 48 = 111$.

4. (Answer: 803)
The garden can be partitioned into regular hexagons congruent to the blocks, and the hexagons on the boundary of the garden form a figure like the path, but with only $n-1$ hexagons on a side.



The figure shows the garden (not including the walk) when $n = 10$. Note that for any $n > 1$, counting the hexagons by columns starting at the left, each column contains one more hexagon than the column adjacent to it on the left, until the center column is reached. Since the leftmost column contains $n-1$ hexagons and the longest column is the $(n-2)^{\text{nd}}$ to the right of it, the longest column contains $(n-1)+(n-2) = 2n-3$ hexagons. The number of hexagons strictly to the left of the center vertical line is therefore

$$(n-1)+n+(n+1)+(n+2)+\cdots+[(n-1)+(n-3)] = \frac{(3n-5)(n-2)}{2},$$

and there are the same number to the right. Since the area of each of the hexagons is six times the area of an equilateral triangle of side 1, the area of the garden is

$$[(2n-3)+(3n-5)(n-2)]\left[\frac{6\sqrt{3}}{4}\right].$$

When $n = 202$ this is $361803\sqrt{3}/2$ square units, so the desired remainder is 803.

**OR**

For $n > 1$, note that when there are $n$ hexagons per side of the bounding path, there is a total of $6(n-1)$ hexagons on the path. The number of hexagons in the interior of the path is

$$1 + 6 + 12 + 18 + \cdots + 6(n-2) = 1 + 3(n-2)(n-1),$$

so when $n = 202$, the garden can be partitioned into $1 + 3(200)(201) = 120601$ hexagons. Since the area of each hexagon is $6\sqrt{3}/4$, the area of the garden is $361803\sqrt{3}/2$.

5. (Answer: 042)
Note that
$$\frac{6^a}{a^6} = \frac{2^a 3^a}{2^{6n} 3^{6m}}$$
is not an integer if and only if $6n > a$ or $6m > a$, that is, if and only if
$$6 \cdot \max(n, m) > 2^n 3^m. \qquad (\star)$$
When both $n \geq 1$ and $m \geq 1$, there are no values of $m$ and $n$ that satisfy $(\star)$. When $m = 0$, $(\star)$ reduces to $2^n < 6n$, which is satisfied only by $n = 1, 2, 3, 4$. When $n = 0$, $(\star)$ reduces to $3^m < 6m$, which is satisfied only by $m = 1, 2$. Thus there are six values of $a$ for which $a^6$ not a divisor of $6^a$, namely, $a = 2, 4, 8, 16, 3$, and 9, and their sum is 42.

6. (Answer: 521)
Because $\dfrac{1}{n^2-4} = \dfrac{1}{4}\left(\dfrac{1}{n-2} - \dfrac{1}{n+2}\right)$, the series telescopes, and it follows that
$$\begin{aligned}
1000 \sum_{n=3}^{10000} \frac{1}{n^2-4} &= 250\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{9999} - \frac{1}{10000} - \frac{1}{10001} - \frac{1}{10002}\right) \\
&= 250 + 125 + \frac{250}{3} + \frac{250}{4} - \frac{250}{9999} - \frac{250}{10000} - \frac{250}{10001} - \frac{250}{10002} \\
&= 520 + \frac{5}{6} - r,
\end{aligned}$$
where the positive number $r$ is less than $1/3$. Thus the requested integer is 521.

7. (Answer: 112)
The sum is a multiple of 200 if and only if $k(k+1)(2k+1) = 6 \cdot 200N = 2^4 \cdot 3 \cdot 5^2 N$ for some positive integer $N$. Because $2k+1$ is odd and $k$ and $k+1$ cannot both be

even, it follows that either $k$ or $k+1$ is a multiple of 16. Furthermore, the product is divisible by 3 for all integer values of $k$. (Why?) Substitute $k = 15, 16, 31, 32, \ldots$, and check whether $k(k+1)(2k+1)$ is divisible by 25 to see that $k = 112$ is the smallest positive integer for which $k(k+1)(2k+1)$ is a multiple of 1200.

8. (Answer: 049)
The equation $\left\lfloor \dfrac{2002}{n} \right\rfloor = k$ is equivalent to

$$k \le \frac{2002}{n} < k+1, \quad \text{or} \quad \frac{2002}{k+1} < n \le \frac{2002}{k}.$$

In order that there be no solutions, there can be no integer in this interval, that is, $\dfrac{2002}{k}$ and $\dfrac{2002}{k+1}$ must have the same integer part. The length of the interval must be less than 1, so

$$\frac{2002}{k} - \frac{2002}{k+1} < 1$$

which yields $k(k+1) > 2002$, and thus $k \ge 45$. For $k = 45, 46, 47, 48, 49, 50$, the integer part of $2002/k$ is $44, 43, 42, 41, 40, 40$, respectively. Thus $2002/49$ and $2002/50$ have the same integer part, so the least positive integer value of $k$ is 49.

9. (Answer: 501)
Let $k$ be the number of elements of $\mathcal{S}$, and let $A$ and $B$ be two empty jars into which elements of $\mathcal{S}$ will be placed to create two disjoint subsets. For each element $x$ in $\mathcal{S}$, there are three possibilities: place $x$ in $A$, place $x$ in $B$, or place $x$ in neither $A$ nor $B$. Thus the number of ordered pairs of disjoint subsets $(A, B)$ is $3^k$. However, this counts the pairs where $A$ or $B$ is empty. Note that for $A$ to be empty, there are two possibilities for each element $x$ in $\mathcal{S}$: place $x$ in $B$, or do not place $x$ in $B$. The number of pairs for which $A$ or $B$ is empty is thus $2^k + 2^k - 1 = 2^{k+1} - 1$. Since interchanging $A$ and $B$ does not yield a different set of subsets, there are $\frac{1}{2}(3^k - 2^{k+1} + 1) = \frac{1}{2}(3^k + 1) - 2^k$ sets. When $k = 10$, $n = \frac{3^{10}+1}{2} - 2^{10} = 28501$, and the desired remainder is 501.

**OR**

For each non-empty subset $X$ of $\mathcal{S}$ with $k$ elements, $k = 1, 2, 3, \ldots, 9$, there are $2^{10-k} - 1$ non-empty subsets of $\mathcal{S}$ that are disjoint with $X$ . Let $N$ be the total number of ordered pairs of non-empty disjoint subsets of $\mathcal{S}$. Then

$$N = \sum_{k=1}^{9} \binom{10}{k} (2^{10-k} - 1) = \sum_{k=1}^{9} \binom{10}{k} (2^k - 1) = \sum_{k=1}^{9} \binom{10}{k} 2^k - \sum_{k=1}^{9} \binom{10}{k}.$$

Note that

$$\sum_{k=0}^{10} \binom{10}{k} = 2^{10},$$

and, from the Binomial Expansion,

$$3^{10} = (1+2)^{10} = \sum_{k=0}^{10} \binom{10}{k} 2^k.$$

Thus $N = \left[3^{10} - (1 + 2^{10})\right] - \left[2^{10} - (1+1)\right] = 3^{10} - 2^{11} + 1$, and the number of sets of two non-empty disjoint subsets of $\mathcal{S}$ is $\frac{1}{2}(3^{10} - 2^{11} + 1) = 28501$.

10. (Answer: 900)

Because $x$ radians is equivalent to $\dfrac{180x}{\pi}$ degrees, the requested special values of $x$ satisfy $\sin x^\circ = \sin \dfrac{180x}{\pi}^\circ$. It follows from properties of the sine function that either

$$\frac{180x}{\pi} = x + 360j \quad \text{or} \quad 180 - \frac{180x}{\pi} = x - 360k$$

for some integers $j$ and $k$. Thus either $x = \dfrac{360j\pi}{180-\pi}$ or $x = \dfrac{180(2k+1)\pi}{180+\pi}$, and the least positive values of $x$ are $\dfrac{360\pi}{180-\pi}$ and $\dfrac{180\pi}{180+\pi}$, so $m+n+p+q = 900$.

11. (Answer: 518)

Let the two series be

$$\sum_{k=0}^{\infty} a \cdot r^k \quad \text{and} \quad \sum_{k=0}^{\infty} b \cdot s^k.$$

The given conditions imply that $a = 1 - r$, $b = 1 - s$, and $ar = bs$. It follows that $r(1-r) = s(1-s)$, that is $r - s = r^2 - s^2$. Because the series are not identical, $r \neq s$, leaving $r = 1 - s$ as the only possibility, and the series may be written as

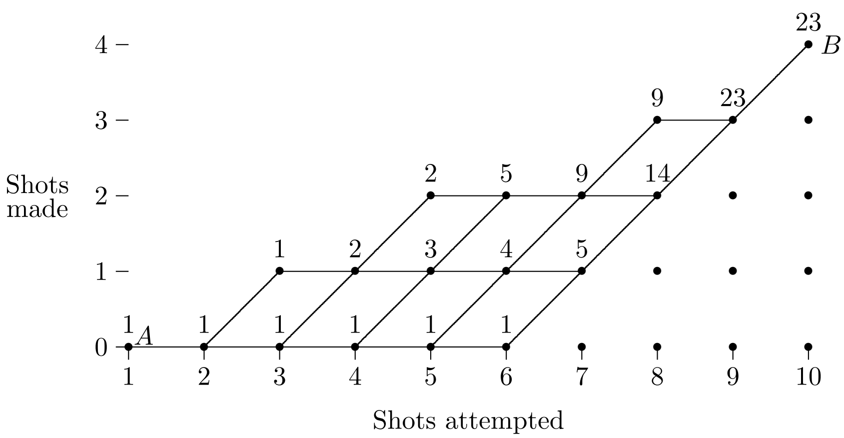
$$\sum_{k=0}^{\infty} (1-r) \cdot r^k \quad \text{and} \quad \sum_{k=0}^{\infty} r \cdot (1-r)^k.$$

As we may pick either series as the one whose third term is $1/8$, set $(1-r)r^2 = 1/8$, from which we obtain $8r^3 - 8r^2 + 1 = 0$. The substitution $t = 2r$ yields $t^3 - 2t^2 + 1 = 0$, for which 1 is a root. Factoring gives $(t-1)(t^2 - t - 1) = 0$, so the other two roots are $(1 \pm \sqrt{5})/2$, which implies that $r = 1/2$ or $r = (1 \pm \sqrt{5})/4$. However, if $r$ were $1/2$, the two series would be equal; and if $r$ were $(1 - \sqrt{5})/4$, then $s$ would be $(3+\sqrt{5})/4$, but convergence requires that $|s|<1$. Thus $r=(1+\sqrt{5})/4$, and $-1<s=(3-\sqrt{5})/4<1$. The second term of the series is therefore equal to

$$r(1-r)=\left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{3-\sqrt{5}}{4}\right)=\frac{\sqrt{5}-1}{8},$$

and $100m+10n+p=518$.

12. (Answer: 660)
Let $x$ be the number of attempts and $y$ the number of shots made. The maximum values of $y$ for $x=1,2,\ldots,10$ are 0, 0, 1, 1, 2, 2, 2, 3, 3, and 4, respectively. Since $y=4$ when $x=10$, the minimum values of $y$ when $x=9,8,7,6,5,4,3,2,1$ are 3, 2, 1, 0, 0, 0, 0, 0, 0, respectively. We can represent the possible sequences of made and missed shots in the diagram below.




The possible sequences of made and attempted shots correspond to sequences of ordered pairs $(x,y)$, where $x$ is the number of shots attempted and $y$ is the number of shots made, beginning at $(1,0)$ and ending at $(10,4)$. Each sequence corresponds to a path from $A$ to $B$ that moves right and/or up on the lines in the diagram. The number of paths from $A$ to any point $P$ on the diagram is the sum of the number of paths from $A$ to the points directly before $P$. Each point is labeled with the number of possible paths from $A$ to that point. Thus, the number of paths from $A$ to $B$ is 23. Each path represents a sequence of 4 made shots and 6 misses, so the requested probability is

$$23(.4)^4(.6)^6=\frac{2^4 3^6 23}{5^{10}},$$

and $(p+q+r+s)(a+b+c)=(2+3+23+5)(4+6+10)=660$.

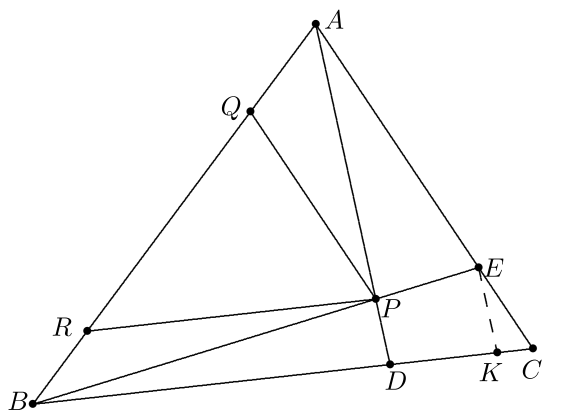
13. (Answer: 901)
Draw the line through $E$ that is parallel to $\overline{AD}$, and let $K$ be its intersection with $\overline{BC}$. Because $CD = 2$ and $KC : KD = EC : EA = 1 : 3$, it follows that $KD = 3/2$. Therefore,

$$\frac{QP}{AE} = \frac{BP}{BE} = \frac{BD}{BK} = \frac{5}{5 + (3/2)} = \frac{10}{13}.$$

Thus

$$\frac{QP}{AC} = \frac{3}{4} \cdot \frac{10}{13} = \frac{15}{26}.$$

Since triangles $PQR$ and $CAB$ are similar, the ratio of their areas is $(15/26)^2 = 225/676$. Thus $m + n = 901$.




**OR**

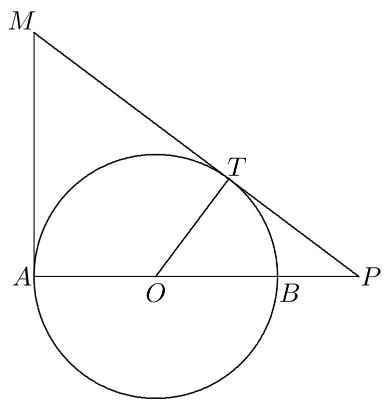
Use *mass points.* Assign a mass of 15 to $C$. Since $AE = 3 \cdot EC$, the mass at $C$ must be 3 times the mass at $A$, so the mass at $A$ is 5, and the mass at $E$ is $15 + 5 = 20$. Similarly, the mass at $B$ is $(2/5) \cdot 15 = 6$, so the mass at $D$ is $15 + 6 = 21$, and the mass at $P$ is $6 + 20 = 26$. Draw $\overrightarrow{CP}$ and let it intersect $\overline{AB}$ at $F$. The mass at $F$ is $26 - 15 = 11$, so $PF/CF = 15/26$, and the ratio of the areas is $(15/26)^2 = 225/676$.

**Remark:** Research *mass points* to find out more about this powerful method of solving problems involving geometric ratios.

14. (Answer: 098)
Let $T$ and $B$ be the points where the circle meets $\overline{PM}$ and $\overline{AP}$, respectively, with $\overline{ABP}$. Triangles $POT$ and $PAM$ are right triangles that share angle $MPA$, so they are similar. Let $p_1$ and $p_2$ be their respective perimeters. Then $OT/AM = p_1/p_2$. Because $AM = TM$, it follows that $p_1 = p_2 - (AM + TM) = 152 - 2AM$. Thus $19/AM = (152 - 2AM)/152$, so that $AM = 38$ and $p_1 = 76$. It is also true that $OP/PM = p_1/p_2$, so

$$\frac{1}{2} = \frac{OP}{PM} = \frac{OP}{152 - (38 + 19 + OP)}$$

It follows that $OP = 95/3$, and $m + n = 98$.




15. (Answer: 282)
Let $r_1$ and $r_2$ be the radii and $A_1$ and $A_2$ be the centers of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and let $P = (u, v)$ belong to both circles. Because the circles have common external tangents that meet at the origin $O$, it follows that the first-quadrant angle formed by the lines $y = 0$ and $y = mx$ is bisected by the ray through $O$, $A_1$, and $A_2$. Therefore, $A_1 = (x_1, kx_1)$ and $A_2 = (x_2, kx_2)$, where $k$ is the tangent of the angle formed by the positive $x$-axis and the ray $OA_1$. Notice that $r_1 = kx_1$ and $r_2 = kx_2$. It follows from the identity $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$ that $m = \dfrac{2k}{1 - k^2}$. Now $(PA_1)^2 = (kx_1)^2$, or

$$(u - x_1)^2 + (v - kx_1)^2 = k^2x_1^2, \text{ so}$$
$$(x_1)^2 - 2(u + kv)x_1 + u^2 + v^2 = 0.$$

In similar fashion, it follows that

$$(x_2)^2 - 2(u + kv)x_2 + u^2 + v^2 = 0.$$

Thus $x_1$ and $x_2$ are the roots of the equation

$$x^2 - 2(u + kv)x + u^2 + v^2 = 0,$$

which implies that $x_1 x_2 = u^2 + v^2$, and that $r_1 r_2 = k^2 x_1 x_2 = k^2(u^2 + v^2)$. Thus

$$k = \sqrt{\frac{r_1 r_2}{u^2 + v^2}}$$

and

$$m = \frac{2k}{1 - k^2} = \frac{2\sqrt{r_1 r_2(u^2 + v^2)}}{u^2 + v^2 - r_1 r_2}.$$

When $u = 9$, $v = 6$, and $r_1 r_2 = 68$, this gives $m = \dfrac{12\sqrt{221}}{49}$, so $a + b + c = 282$.

Problem Authors

1. *David Hankin*
2. *Rick Parris*
3. *Susan Wildstrom*
4. *Leo Schneider*
5. *Franz Rothe*
6. *Leo Schneider*
7. *Harold Reiter*
8. *David Hankin*
9. *Chris Jeuell*
10. *Rick Parris*
11. *Kent Boklan*
12. *Steven Blasberg*
13. *Sam Baethge*
14. *Krassimir Penev*
15. *Zuming Feng*

*The*

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