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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, April 8, 2003

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 336)

   Call the three integers a, b, and c, and, without loss of generality, assume $a \leq b \leq c$. Then $abc = 6(a+b+c)$, and $c = a+b$. Thus $abc = 12c$, and $ab = 12$, so $(a, b, c) = (1, 12, 13)$, $(2, 6, 8)$, or $(3, 4, 7)$, and $N = 156$, 96, or 84. The sum of the possible values of N is 336.

2. (Answer: 120)

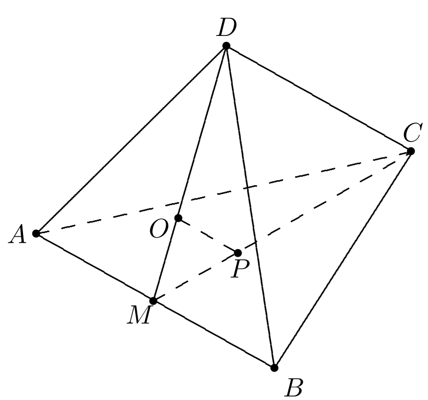
   An integer is divisible by 8 if and only if the number formed by the rightmost three digits is divisible by 8. The greatest integer with the desired property is formed by choosing 9876543 as the seven leftmost digits and finding the arrangement of 012 that yields the greatest multiple of 8, assuming that such an arrangement exists. Checking the 6 permutations of 012 yields 120 as the sole multiple of 8, so $N = 9876543120$, and its remainder when divided by 1000 is 120.

3. (Answer: 192)

   There are three choices for the first letter and two choices for each subsequent letter, so there are $3 \cdot 2^{n-1}$ n-letter good words. Substitute $n = 7$ to find there are $3 \cdot 2^6 = 192$ seven-letter good words.

4. (Answer: 028)

   Let O and P be the centers of faces DAB and ABC, respectively, of regular tetrahedron $ABCD$. Both $\overrightarrow{DO}$ and $\overrightarrow{CP}$ intersect $\overline{AB}$ at its midpoint M. Since $\dfrac{MO}{MD} = \dfrac{MP}{MC} = \dfrac{1}{3}$, triangles MOP and MDC are similar, and $OP = (1/3)DC$. Because the tetrahedra are similar, the ratio of their volumes is the cube of the ratio of a pair of corresponding sides, namely, $(1/3)^3 = 1/27$, so $m + n = 28$.



5. (Answer: 216)

Let $\overline{AB}$ be a diameter of the circular face of the wedge formed by the first cut, and let $\overline{AC}$ be the longest chord across the elliptical face of the wedge formed by the second cut. Then $\triangle ABC$ is an isosceles right triangle and $BC = 12$ inches. If a third cut were made through the point C on the log and perpendicular to the axis of the cylinder, then a second wedge, congruent to the original, would be formed, and the two wedges would fit together to form a right circular cylinder with radius $AB/2 = 6$ inches and height BC. Thus, the volume of the wedge is $\frac{1}{2}\pi \cdot 6^2 \cdot 12 = 216\pi$, and $n = 216$.

6. (Answer: 112)

Let M be the midpoint of $\overline{BC}$, let M' be the reflection of M in G, and let Q and R be the points where $\overline{BC}$ meets $\overline{A'C'}$ and $\overline{A'B'}$, respectively. Note that since M is on $\overline{BC}$, M' is on $\overline{B'C'}$. Because a 180° rotation maps each line that does not contain the center of the rotation to a parallel line, $\overline{BC}$ is parallel to $\overline{B'C'}$, and $\triangle A'RQ$ is similar to $\triangle A'B'C'$. Recall that medians of a triangle trisect each other to obtain

$$M'G = MG = (1/3)AM, \text{ so } A'M = AM' = (1/3)AM = (1/3)A'M'.$$

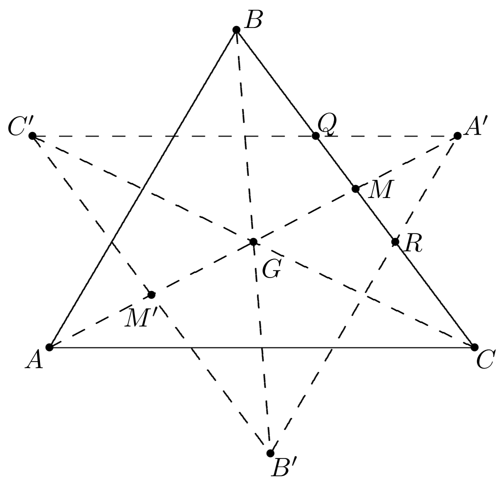
Thus the similarity ratio between triangles $A'RQ$ and $A'B'C'$ is 1/3, and

$$[A'RQ] = (1/9)[A'B'C'] = (1/9)[ABC].$$

Similarly, the area of each of the two small triangles with vertices at B' and C', respectively, is 1/9 that of $\triangle ABC$. The desired area is therefore

$$[ABC] + 3(1/9)[ABC] = (4/3)[ABC].$$

Use Heron's formula, $K = \sqrt{s(s-a)(s-b)(s-c)}$, to find $[ABC] = \sqrt{21 \cdot 7 \cdot 6 \cdot 8} = 84$. The desired area is then $(4/3) \cdot 84 = 112$.



7. (Answer: 400)

Let O be the point of intersection of diagonals $\overline{AC}$ and $\overline{BD}$, and E the point of intersection of $\overline{AC}$ and the circumcircle of $\triangle ABD$. Extend $\overline{DB}$ to meet the circumcircle of $\triangle ACD$ at F. From the Power-of-a-Point Theorem, we have

$$AO \cdot OE = BO \cdot OD \text{ and } DO \cdot OF = AO \cdot OC.$$

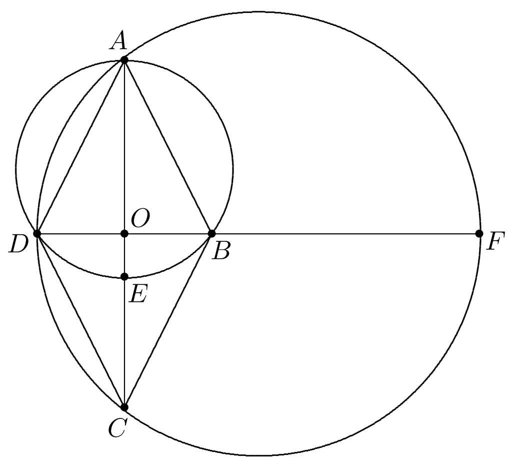
Let $AC = 2m$ and $BD = 2n$. Because $\overline{AE}$ is a diameter of the circumcircle of $\triangle ABD$, and $\overline{DF}$ is a diameter of the circumcircle of $\triangle ACD$, the above equalities can be rewritten as

$$m(25 - m) = n^2 \quad \text{and} \quad n(50 - n) = m^2,$$

or

$$25m = m^2 + n^2 \quad \text{and} \quad 50n = m^2 + n^2.$$

Therefore $m = 2n$. It follows that $50n = 5n^2$, so $n = 10$ and $m = 20$. Thus $[ABCD] = (1/2)AC \cdot BD = 2mn = 400$.



OR

Let R_1 and R_2 be the circumradii of triangles ABD and ACD, respectively. Because $\overline{BO}$ is the altitude to the hypotenuse of right $\triangle ABE$, $AB^2 = AO \cdot AE$. Similarly, in right $\triangle DAF$, $AB^2 = DA^2 = DO \cdot DF$, so $AO \cdot AE = DO \cdot DF$. Thus

$$\frac{AO}{DO} = \frac{DF}{AE} = \frac{R_2}{R_1} = 2.$$

Also, from right $\triangle ADE$, $2 = \dfrac{AO}{DO} = \dfrac{DO}{OE}$. Then

$$25 = 2R_1 = AE = AO + OE = 2 \cdot DO + \frac{1}{2}DO = \frac{5}{2}DO,$$

and $DO = 10$, $AO = 20$, so $[ABCD] = 400$.

OR

Let s be the length of a side of the rhombus, and let $\alpha = \angle BAC$. Then $AO = s\cos\alpha$, and $BO = s\sin\alpha$, so $[ABCD] = 4[ABO] = 2s^2\sin\alpha\cos\alpha = s^2\sin 2\alpha$. Apply the Extended Law of Sines (In any $\triangle ABC$ with $AB = c$, $BC = a$, $CA = b$, and circumradius R, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$) in $\triangle ABD$ and $\triangle ACD$ to obtain $s = 2R_1\sin(90^\circ - \alpha) = 2R_1\cos\alpha$, and $s = 2R_2\sin\alpha$. Thus $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{R_1}{R_2} = \dfrac{1}{2}$. Also, $s^2 = 4R_1R_2\cos\alpha\sin\alpha = 2R_1R_2\sin 2\alpha$. But $\sin 2\alpha = 2\cdot\dfrac{1}{\sqrt{5}}\cdot\dfrac{2}{\sqrt{5}} = \dfrac{4}{5}$, from which $[ABCD] = 2R_1R_2\sin^2 2\alpha = 2\cdot\frac{25}{2}\cdot 25\cdot\frac{16}{25} = 400$.

OR

Let $AB = s$, $AO = m$, and $BO = n$, and use the fact that the product of the lengths of the sides of a triangle is four times the product of its area and its circumradius to obtain $4[ABD]R_1 = s \cdot s \cdot 2n$ and $4[ACD]R_2 = s \cdot s \cdot 2m$. Since $[ABD] = [ACD]$, conclude that $\dfrac{1}{2} = \dfrac{R_1}{R_2} = \dfrac{n}{m}$, and proceed as above.

8. (Answer: 348)

The nth term of an arithmetic sequence has the form $a_n = pn+q$, so the product of corresponding terms of two arithmetic sequences is a quadratic expression, $s_n = an^2 + bn + c$. Letting $n = 0$, 1, and 2 produces the equations $c = 1440$, $a + b + c = 1716$, and $4a + 2b + c = 1848$, whose common solution is $a = -72$, $b = 348$, and $c = 1440$. Thus the eighth term is $s_7 = -72\cdot 7^2 + 348\cdot 7 + 1440 = 348$. Note that $s_n = -72n^2 + 348n + 1440 = -12(2n - 15)(3n + 8)$ can be used to generate pairs of arithmetic sequences with the desired products, such as $\{180, 156, 132, \ldots\}$ and $\{8, 11, 14, \ldots\}$.

9. (Answer: 006)

Apply the division algorithm for polynomials to obtain

$$P(x) = Q(x)(x^2 + 1) + x^2 - x + 1.$$

Therefore

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} z_i^2 - \sum_{i=1}^{4} z_i + 4 = \left(\sum_{i=1}^{4} z_i\right)^2 - 2\sum_{i<j} z_i z_j - \sum_{i=1}^{4} z_i + 4.$$

Use the formulas for sum and product of the roots to obtain $\sum_{i=1}^{4} P(z_i) = 1 + 2 - 1 + 4 = 6$.

OR

Since, for each root w of $Q(x) = 0$, we have $w^4 - w^3 - w^2 - 1 = 0$, conclude that $w^4 - w^3 = w^2 + 1$, and then $w^6 - w^5 = w^4 + w^2 = w^3 + 2w^2 + 1$. Thus $P(w) = w^3 + 2w^2 + 1 - w^3 - w^2 - w = w^2 - w + 1$. Therefore

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} z_i^2 - \sum_{i=1}^{4} z_i + 4,$$

and, as above, $\sum_{i=1}^{4} P(z_i) = 6$.

10. (Answer: 156)

Let x represent the smaller of the two integers. Then $\sqrt{x} + \sqrt{x+60} = \sqrt{y}$, and $x + x + 60 + 2\sqrt{x(x+60)} = y$. Thus $x(x+60) = z^2$ for some positive integer z. It follows that

$$\begin{aligned} x^2 + 60x &= z^2, \\ x^2 + 60x + 900 &= z^2 + 900, \\ (x+30)^2 - z^2 &= 900, \quad \text{and} \\ (x + 30 + z)(x + 30 - z) &= 900. \end{aligned}$$

Thus $(x+30+z)$ and $(x+30-z)$ are factors of 900 with $(x+30+z) > (x+30-z)$, and they are both even because their sum and product are even. Note that each pair of even factors of 900 can be found by doubling factor-pairs of 225, so the possible values of $(x+30+z, x+30-z)$ are $(450, 2)$, $(150, 6)$, $(90, 10)$, and $(50, 18)$. Each of these pairs yields a value for x which is 30 less than half their sum. These values are 196, 48, 20, and 4. When $x = 196$ or 4, then $\sqrt{x}+\sqrt{x+60}$ is an integer. When $x = 48$, we obtain $\sqrt{48}+\sqrt{108} = \sqrt{300}$, and when $x = 20$, we obtain $\sqrt{20}+\sqrt{80} = \sqrt{180}$. Thus the desired maximum sum is $48 + 108 = 156$.

OR

Let x represent the smaller of the two integers. Then $\sqrt{x}+\sqrt{x+60} = \sqrt{y}$, and $x + x + 60 + 2\sqrt{x(x+60)} = y$. Thus $x(x + 60) = z^2$ for some positive integer z. Let d be the greatest common divisor of x and $x + 60$. Then $x = dm$ and $x + 60 = dn$, where m and n are relatively prime. Because $dm \cdot dn = z^2$, there are relatively prime positive integers p and q such that $m = p^2$ and $n = q^2$. Now $d(q^2 - p^2) = 60$. Note that p and q cannot both be odd, else $q^2 - p^2$ would be divisible by 8; and they cannot both be even because they are relatively prime. Therefore p and q are of opposite parity, and $q^2 - p^2$ is odd, which implies that $q^2 - p^2 = 1, 3, 5$, or 15. But $q^2 - p^2$ cannot be 1, and if $q^2 - p^2$ were 15, then d would be 4, and x and $x + 60$ would be squares. Thus $q^2 - p^2 = 3$ or 5, and $(q+p, q-p) = (3, 1)$ or $(5, 1)$, and then $(q, p) = (2, 1)$ or $(3, 2)$. This yields $(x+60, x) = (2^2 \cdot 20, 1^2 \cdot 20) = (80, 20)$ or $(x+60, x) = (3^2 \cdot 12, 2^2 \cdot 12) = (108, 48)$, so the requested maximum sum is $108 + 48 = 156$.

OR

Let a and b represent the two integers, with $a > b$. Then $a - b = 60$, and $\sqrt{a}+\sqrt{b} = \sqrt{c}$, where c is an integer that is not a square. Dividing yields $\sqrt{a}-\sqrt{b} = 60/\sqrt{c}$. Adding these last two equations yields

$$2\sqrt{a} = \sqrt{c} + \frac{60}{\sqrt{c}}, \quad \text{so}$$
$$2\sqrt{ac} = c + 60.$$

Therefore $\sqrt{ac}$ is an integer, so c is even, as is ac, which implies $\sqrt{ac}$ is even. Hence c is a multiple of 4, so there is a positive non-square integer d such that $c = 4d$. Then

$$a = \frac{(c+60)^2}{4c} = \frac{(4d+60)^2}{16d} = \frac{(d+15)^2}{d} = \frac{d^2+30d+225}{d} = d + \frac{225}{d} + 30.$$

Thus d is a non-square divisor of 225, so the possible values of d are 3, 5, 15, 45, and 75. The maximum value of a, which occurs when $d = 3$ or $d = 75$, is

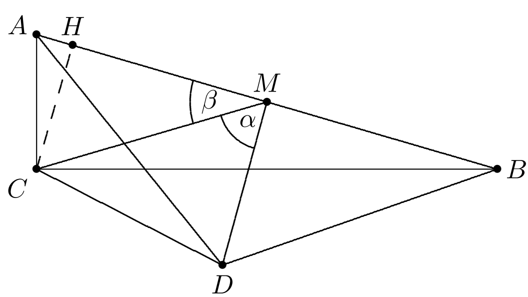
$3+75+30 = 108$, so the maximum value of b is $108-60 = 48$, and the requested maximum sum is $48+108 = 156$.

11. (Answer: 578)

The desired area is given by $(1/2)\cdot CM \cdot DM \cdot \sin\alpha$, where $\alpha = \angle CMD$. The length $AB = \sqrt{7^2+24^2} = 25$, and, since $\overline{CM}$ is the median to the hypotenuse of $\triangle ABC$, $CM = 25/2$. Because $\overline{DM}$ is both the altitude and median to side AB in $\triangle ABD$, $DM = 5\sqrt{11}/2$ by the Pythagorean Theorem. To compute $\sin\alpha$, let $\angle AMC = \beta$, and note that $\angle AMC$ and $\angle CMD$ are complementary, so $\cos\beta = \sin\alpha$. Apply the Law of Cosines in $\triangle AMC$ to obtain

$$\cos\beta = \frac{(\frac{25}{2})^2 + (\frac{25}{2})^2 - 7^2}{2\cdot\frac{25}{2}\cdot\frac{25}{2}} = \frac{527}{625}.$$

The area of $\triangle CMD$ is $(1/2)\cdot CM\cdot DM\cdot\sin\alpha = \frac{1}{2}\cdot\frac{25}{2}\cdot\frac{5\sqrt{11}}{2}\cdot\frac{527}{625} = \frac{527\sqrt{11}}{40}$, and $m+n+p = 527+11+40 = 578$.



OR

Let $\overline{CH}$ be the altitude to hypotenuse $\overline{AB}$. Triangles CDM and HDM share side $\overline{DM}$, and because $\overline{DM} \parallel \overline{CH}$, $[CDM] = [HDM] = (1/2)HM\cdot DM$. Note that $DM = \sqrt{AD^2 - AM^2} = \frac{5\sqrt{11}}{2}$, and that $AC^2 = AH\cdot AB$. Then $AH = 49/25$, and $HM = (25/2)-(49/25) = 527/50$. Thus $[CDM] = \frac{1}{2}\cdot\frac{527}{50}\cdot\frac{5\sqrt{11}}{2} = \frac{527}{40}\sqrt{11}$.

OR

Denote the vector $\overrightarrow{CD}$ by $\vec{d}$ and the vector $\overrightarrow{CM}$ by $\vec{m}$. Then $\vec{m} = (12, 7/2, 0)$, from which $\vec{d} = \big(12-(7/25)k, 7/2-(24/25)k, 0\big)$, where $k = (5\sqrt{11})/2$. This can be simplified to obtain $\vec{d} = \big(12-(7\sqrt{11}/10), 7/2-(24\sqrt{11}/10), 0\big)$. The area of $\triangle CDM$ is therefore $(1/2)|\vec{m}\times\vec{d}| = (527/40)\sqrt{11}$.

12. (Answer: 134)

Let t be the number of members of the committee, n_k be the number of votes for candidate k, and let p_k be the percentage of votes for candidate k for $k = 1, 2, \ldots, 27$. We have

$$n_k \geq p_k + 1 = \frac{100n_k}{t} + 1.$$

Adding these 27 inequalities yields $t \geq 127$. Solving for n_k gives $n_k \geq \dfrac{t}{t-100}$, and, since n_k is an integer, we obtain

$$n_k \geq \left\lceil \frac{t}{t-100} \right\rceil,$$

where the notation $\lceil x \rceil$ denotes the least integer that is greater than or equal to x. The last inequality is satisfied for all $k = 1, 2, \ldots, 27$ if and only if it is satisfied by the smallest n_k, say n_1. Since $t \geq 27n_1$, we obtain

$$t \geq 27 \left\lceil \frac{t}{t-100} \right\rceil \qquad (1)$$

and our problem reduces to finding the smallest possible integer $t \geq 127$ that satisfies the inequality (1). If $\frac{t}{t-100} > 4$, that is, $t \leq 133$, then $27 \left\lceil \frac{t}{t-100} \right\rceil \geq 27 \cdot 5 = 135$ so that the inequality (1) is not satisfied. Thus 134 is the least possible number of members in the committee. Note that when $t = 134$, an election in which 1 candidate receives 30 votes and the remaining 26 candidates receive 4 votes each satisfies the conditions of the problem.

OR

Let t be the number of members of the committee, and let m be the least number of votes that any candidate received. It is clear that $m \neq 0$ and $m \neq 1$. If $m = 2$, then $2 \geq 1 + 100(2/t)$, so $t \geq 200$. Similarly, if $m = 3$, then $3 \geq 1 + 100(3/t)$, and $t \geq 150$; and if $m = 4$, then $4 \geq 1 + 100(4/t)$, so $t \geq 134$. When $m \geq 5$, $t \geq 27 \cdot 5 = 135$. Thus $t \geq 134$. Verify that t can be 134 by noting that the votes may be distributed so that 1 candidate receives 30 votes and the remaining 26 candidates receive 4 votes each.

13. (Answer: 683)

If the bug is at the starting vertex after move n, the probability is 1 that it will move to a non-starting vertex on move $n + 1$. If the bug is not at the starting vertex after move n, the probability is $1/2$ that it will move back to its starting vertex on move $n+1$, and the probability is $1/2$ that it will move to another non-starting starting vertex on move $n+1$. Let p_n be the probability that the bug is at the starting vertex after move n. Then $p_{n+1} = 0 \cdot p_n + \frac{1}{2}(1 - p_n) = -\frac{1}{2}p_n + \frac{1}{2}$. This implies that $p_{n+1} - \frac{1}{3} = -\frac{1}{2}(p_n - \frac{1}{3})$. Since $p_0 - \frac{1}{3} = 1 - \frac{1}{3} = 2/3$, conclude that $p_n - \frac{1}{3} = \frac{2}{3} \cdot (-\frac{1}{2})^n$. Therefore

$$p_n = \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^n + \frac{1}{3} = \frac{1 + (-1)^n \frac{1}{2^{n-1}}}{3} = \frac{2^{n-1} + (-1)^n}{3 \cdot 2^{n-1}}.$$

Substitute 10 for n to find that $p_{10} = 171/512$, and $m + n$ is 683.

OR

A 10-step path can be represented by a 10-letter sequence consisting of only A's and B's, where A represents a move in the clockwise direction and B represents a move in the counterclockwise direction. Where the path ends depends on the number of A's and B's, not on their arrangement. Let x be the number of A's, and let y be the number of B's. Note that the bug will be home if and only if $x - y$ is a multiple of 3. After 10 moves, $x + y = 10$. Then $2x = 10 + 3k$ for some integer k, and so $x = 5 + 3j$ for some integer j. Thus the number of A's must be 2, 5, or 8, and the desired probability is

$$\frac{\binom{10}{2} + \binom{10}{5} + \binom{10}{8}}{2^{10}} = \frac{171}{512}.$$

OR

Let X be the bug's starting vertex, and let Y and Z be the other two vertices. Let x_k, y_k, and z_k be the probabilities that the bug is at vertex X, Y, and Z, respectively, at move k, for $k \geq 0$. Then $x_{k+1} = .5y_k + .5z_k$, $y_{k+1} = .5x_k + .5z_k$, and $z_{k+1} = .5x_k + .5y_k$. This can be written as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_{10} \\ y_{10} \\ z_{10} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix}^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $x_{10} = 171/512$.

14. (Answer: 051)

Let the x-coordinates of C, D, E, and F be c, d, e, and f, respectively. Note that the y-coordinate of C is not 4, since, if it were, the fact that $AB = BC$ would imply that A, B, and C are collinear or that c is 0. Therefore $F = (f, 4)$. Since $\overline{AF}$ and $\overline{CD}$ are both parallel and congruent, $C = (c, 6)$ and $D = (d, 10)$, and then $E = (e, 8)$. Because the y-coordinates of B, C, and D are 2, 6, and 10, respectively, and $BC = CD$, conclude that $b = d$. Since $\overline{AB}$ and $\overline{DE}$ are both parallel and congruent, $e = 0$. Let a denote the side-length of the hexagon. Then $f^2+16 = AF^2 = a^2 = AB^2 = b^2+4$. Apply the Law of Cosines in $\triangle ABF$ to obtain $3a^2 = BF^2 = (b-f)^2 + 4$. Without loss of generality, assume $b > 0$. Then $f < 0$ and $b = \sqrt{a^2-4}$, $f = -\sqrt{a^2-16}$, and $b - f = \sqrt{3a^2-4}$. Now

$$\begin{aligned} \sqrt{a^2-4} + \sqrt{a^2-16} &= \sqrt{3a^2-4}, \quad \text{so} \\ 2a^2 - 20 + 2\sqrt{(a^2-4)(a^2-16)} &= 3a^2 - 4, \quad \text{and} \\ 2\sqrt{(a^2-4)(a^2-16)} &= a^2 + 16. \end{aligned}$$

Squaring again and simplifying yields $a^2 = 112/3$, so $b = 10/\sqrt{3}$ and $f = -8/\sqrt{3}$. Hence $A = (0,0)$, $B = (10/\sqrt{3}, 2)$, $C = (6\sqrt{3}, 6)$, $D = (10/\sqrt{3}, 10)$, $E = (0, 8)$, $F = (-8/\sqrt{3}, 4)$. Thus $[ABCDEF] = [ABDE] + 2[AEF] = b \cdot AE + (-f) \cdot AE = 8(b-f) = 48\sqrt{3}$, so $m + n = 51$.

OR

Let α denote the measure of the acute angle formed by $\overline{AB}$ and the x-axis. Then the measure of the acute angle formed by $\overline{AF}$ and the x-axis is $60° - \alpha$. Note that $a \sin\alpha = 2$, so

$$\begin{aligned} 4 &= a \sin(60° - \alpha) \\ &= a\frac{\sqrt{3}}{2}\cos\alpha - a \cdot \frac{1}{2}\sin\alpha \\ &= a\frac{\sqrt{3}}{2}\cos\alpha - 1. \end{aligned}$$

Thus $a\sqrt{3}\cos\alpha = 10$, and $b = a\cos\alpha = 10/\sqrt{3}$. Then $a^2 = b^2 + 4 = 112/3$, and $f^2 = a^2 - 16 = 64/3$. Also, $(c-b)^2 = a^2 - 16 = 64/3$, so $c = b + 8/\sqrt{3} = 6\sqrt{3}$; $c - d = 0 - f = 8/\sqrt{3}$, so $d = 10/\sqrt{3}$; and $e - f = c - b = 8/\sqrt{3}$, so $e = 0$. Proceed as above to obtain $[ABCDEF] = 48\sqrt{3}$.

15. (Answer: 015)

Note that

$$P(x) = x + 2x^2 + 3x^3 + \cdots + 24x^{24} + 23x^{25} + 22x^{26} + \cdots + 2x^{46} + x^{47},$$

and

$$xP(x) = \quad x^2 + 2x^3 + 3x^4 + \cdots \quad + 24x^{25} + 23x^{26} + \quad \cdots + \quad 2x^{47} + x^{48},$$

so

$$\begin{aligned}(1-x)P(x) &= x + x^2 + \cdots + x^{24} - (x^{25} + x^{26} + \cdots + x^{47} + x^{48}) \\ &= (1 - x^{24})(x + x^2 + \cdots + x^{24}).\end{aligned}$$

Then, for $x \neq 1$,

$$\begin{aligned}P(x) &= \frac{x^{24}-1}{x-1}\, x\,(1 + x + \cdots + x^{23}) \\ &= x\Big(\frac{x^{24}-1}{x-1}\Big)^2 . \qquad (*)\end{aligned}$$

One zero of $P(x)$ is 0, which does not contribute to the requested sum. The remaining zeros of $P(x)$ are the same as those of $(x^{24}-1)^2$, excluding 1. Because $(x^{24}-1)^2$ and $x^{24}-1$ have the same distinct zeros, the remaining zeros of $P(x)$ can be expressed as $z_k = \operatorname{cis} 15k^\circ$ for $k = 1, 2, 3, \cdots, 23$. The squares of the zeros are therefore of the form $\operatorname{cis} 30k^\circ$, and the requested sum is

$$\sum_{k=1}^{23} |\sin 30k^\circ| = 4\sum_{k=1}^{5} |\sin 30k^\circ| = 4\left(2\cdot(1/2) + 2\cdot(\sqrt{3}/2) + 1\right) = 8 + 4\sqrt{3}.$$

Thus $m + n + p = 15$.

Note: the expression $(*)$ can also be obtained using the identity

$$(1+x+x^2+\cdots+x^n)^2 = 1+2x+3x^2+\cdots+(n+1)x^n+\cdots+3x^{2n-2}+2x^{2n-1}+x^{2n}.$$

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