Mathematical Association of America
American Mathematics Competitions

22nd Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

Tuesday, April 6, 2004

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and geometers are not permitted.**

4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given in your school on TUESDAY and WEDNESDAY, April 27 & 28, 2004.

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $\frac{a\pi + b\sqrt{c}}{d\pi - e\sqrt{f}}$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $a \cdot b \cdot c \cdot d \cdot e \cdot f$ is divided by 1000.

2. A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Given that the probability that they get the same color combination, irrespective of order, is m/n, where m and n are relatively prime positive integers, find $m + n$.

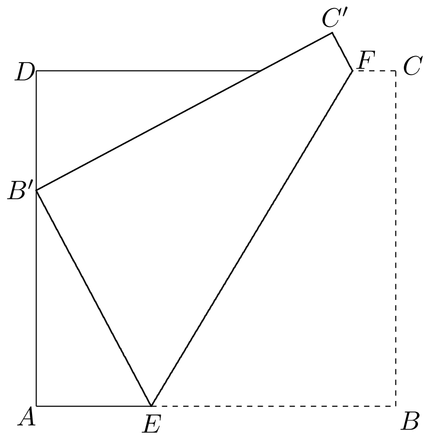
3. A solid rectangular block is formed by gluing together N congruent 1-cm cubes face to face. When the block is viewed so that three of its faces are visible, exactly 231 of the 1-cm cubes cannot be seen. Find the smallest possible value of N.

4. How many positive integers less than 10,000 have at most two different digits?

5. In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before?

6. Three clever monkeys divide a pile of bananas. The first monkey takes some bananas from the pile, keeps three-fourths of them, and divides the rest equally between the other two. The second monkey takes some bananas from the pile, keeps one-fourth of them, and divides the rest equally between the other two. The third monkey takes the remaining bananas from the pile, keeps one-twelfth of them, and divides the rest equally between the other two. Given that each monkey receives a whole number of bananas whenever the bananas are divided, and the numbers of bananas the first, second, and third monkeys have at the end of the process are in the ratio $3 : 2 : 1$, what is the least possible total for the number of bananas?

7. $ABCD$ is a rectangular sheet of paper that has been folded so that corner B is matched with point B' on edge $\overline{AD}$. The crease is $\overline{EF}$, where E is on $\overline{AB}$ and F is on $\overline{CD}$. The dimensions $AE = 8$, $BE = 17$, and $CF = 3$ are given. The perimeter of rectangle $ABCD$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.



8. How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

9. A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms a_{2n-1}, a_{2n}, and a_{2n+1} are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

10. Let $\mathcal{S}$ be the set of integers between 1 and 2^{40} whose binary expansions have exactly two 1's. If a number is chosen at random from $\mathcal{S}$, the probability that it is divisible by 9 is p/q, where p and q are relatively prime positive integers. Find $p + q$.

11. A right circular cone has a base with radius 600 and height $200\sqrt{7}$. A fly starts at a point on the surface of the cone whose distance from the vertex of the cone is 125, and crawls along the surface of the cone to a point on the exact opposite side of the cone whose distance from the vertex is $375\sqrt{2}$. Find the least distance that the fly could have crawled.

12. Let $ABCD$ be an isosceles trapezoid, whose dimensions are $AB = 6$, $BC = 5 = DA$, and $CD = 4$. Draw circles of radius 3 centered at A and B, and circles of radius 2 centered at C and D. A circle contained within the trapezoid is tangent to all four of these circles. Its radius is $\dfrac{-k + m\sqrt{n}}{p}$, where k, m, n, and p are positive integers, n is not divisible by the square of any prime, and k and p are relatively prime. Find $k + m + n + p$.

13. Let $ABCDE$ be a convex pentagon with $\overline{AB} \parallel \overline{CE}$, $\overline{BC} \parallel \overline{AD}$, $\overline{AC} \parallel \overline{DE}$, $\angle ABC = 120^\circ$, $AB = 3$, $BC = 5$, and $DE = 15$. Given that the ratio between the area of $\triangle ABC$ and the area of $\triangle EBD$ is m/n, where m and n are relatively prime positive integers, find $m + n$.

14. Consider a string of n 7's, $\quad 7 \quad 7 \quad 7 \quad 7 \quad \ldots \quad 7 \quad 7, \quad$ into which $+$ signs are inserted to produce an arithmetic expression. For example, $7+77+777+7+7 = 875$ could be obtained from eight 7's in this way. For how many values of n is it possible to insert $+$ signs so that the resulting expression has value 7000?

15. A long thin strip of paper is 1024 units in length, 1 unit in width, and is divided into 1024 unit squares. The paper is folded in half repeatedly. For the first fold, the right end of the paper is folded over to coincide with and lie on top of the left end. The result is a 512 by 1 strip of double thickness. Next, the right end of this strip is folded over to coincide with and lie on top of the left end, resulting in a 256 by 1 strip of quadruple thickness. This process is repeated 8 more times. After the last fold, the strip has become a stack of 1024 unit squares. How many of these squares lie below the square that was originally the 942nd square counting from the left?

Your Exam Manager will have a copy of the 2004 AIME Solution Pamphlet.

WRITE TO US:

Correspondence about the problems and solutions should be addressed to:

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2004 USAMO

THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday, April 27 & Wednesday, April 28. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS

For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.html

MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm

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