The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

55th Annual American Mathematics Contest 12

AMC 12 - Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 10, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
American Mathematics Competitions
University of Nebraska, P.O. Box 81606
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1. **(E)** Since \$20 is 2000 cents, she pays $(0.0145)(2000) = 29$ cents per hour in local taxes.

2. **(C)** The 8 unanswered problems are worth $(2.5)(8) = 20$ points, so Charlyn must earn at least 80 additional points. The smallest multiple of 6 that is at least 80 is $(6)(14) = 84$, so Charlyn must have at least 14 correct answers.

3. **(B)** The value of $x = 100 - 2y$ is a positive integer for each positive integer y with $1 \le y \le 49$.

4. **(E)** Bertha has $30 - 6 = 24$ granddaughters, none of whom have any daughters. The granddaughters are the children of $24/6 = 4$ of Bertha's daughters, so the number of women having no daughters is $30 - 4 = 26$.

5. **(B)** The y-intercept of the line is between 0 and 1, so $0 < b < 1$. The slope is between -1 and 0, so $-1 < m < 0$. Thus $-1 < mb < 0$.

6. **(A)** Since none of V, W, X, Y, or Z exceeds 2004^{2005}, the difference $U - V = 2004^{2005}$ is the largest.

7. **(B)** After three rounds the players A, B, and C have 14, 13, and 12 tokens, respectively. Every subsequent three rounds of play reduces each player's supply of tokens by one. After 36 rounds they have 3, 2, and 1 token, respectively, and after the 37^{th} round Player A has no tokens.

8. **(B)** Let x, y, and z be the areas of $\triangle ADE$, $\triangle BDC$, and $\triangle ABD$, respectively. The area of $\triangle ABE$ is $(1/2)(4)(8) = 16 = x + z$, and the area of $\triangle BAC$ is $(1/2)(4)(6) = 12 = y + z$. The requested difference is

$$x - y = (x + z) - (y + z) = 16 - 12 = 4.$$

9. **(C)** Let r, h, and V, respectively, be the radius, height, and volume of the jar that is currently being used. The new jar will have a radius of $1.25r$ and volume V. Let H be the height of the new jar. Then

$$\pi r^2 h = V = \pi(1.25r)^2 H, \quad \text{so} \quad \frac{H}{h} = \frac{1}{(1.25)^2} = 0.64.$$

Thus H is 64% of h, so the height must be reduced by $(100 - 64)\% = 36\%$.

OR

Multiplying the diameter by $5/4$ multiplies the area of the base by $(5/4)^2 = 25/16$, so in order to keep the same volume, the height must be multiplied by $16/25$. Thus the height must be decreased by $9/25$, or 36%.

10. **(C)** The sum of a set of integers is the product of the mean and the number of integers, and the median of a set of consecutive integers is the same as the mean. So the median must be $7^5/49 = 7^3$.

11. **(A)** If n is the number of coins in Paula's purse, then their total value is $20n$ cents. If she had one more quarter, she would have $n+1$ coins whose total value in cents could be expressed both as $20n+25$ and as $21(n+1)$. Therefore

$$20n + 25 = 21(n+1), \quad \text{so} \quad n = 4.$$

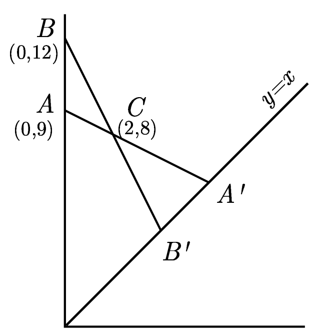
Since Paula has four coins with a total value of 80 cents, she must have three quarters and one nickel, so the number of dimes is 0.

12. **(B)** Line AC has slope $-\frac{1}{2}$ and y-intercept (0,9), so its equation is

$$y = -\frac{1}{2}x + 9.$$

Since the coordinates of A' satisfy both this equation and $y = x$, it follows that $A' = (6,6)$. Similarly, line BC has equation $y = -2x + 12$, and $B' = (4,4)$. Thus

$$A'B' = \sqrt{(6-4)^2 + (6-4)^2} = 2\sqrt{2}.$$



13. **(B)** There are $\binom{9}{2} = 36$ pairs of points in S, and each pair determines a line. However, there are three horizontal, three vertical, and two diagonal lines that pass through three points of S, and these lines are each determined by three different pairs of points in S. Thus the number of distinct lines is $36 - 2 \cdot 8 = 20$.

OR

There are 3 vertical lines, 3 horizontal lines, 3 each with slopes 1 and -1, and 2 each with slopes 2, -2, $1/2$, and $-1/2$, for a total of 20.

14. **(A)** The terms of the arithmetic progression are 9, $9+d$, and $9+2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

$$(11+d)^2 = 9(29+2d) \quad \text{so} \quad d^2 + 4d - 140 = 0.$$

Thus $d = 10$ or $d = -14$. The corresponding geometric progressions are 9, 21, 49 and 9, -3, 1, so the smallest possible value for the third term of the geometric progression is 1.

15. **(C)** When they first meet, they have run a combined distance equal to half the length of the track. Between their first and second meetings, they run a combined distance equal to the full length of the track. Because Brenda runs at a constant speed and runs 100 meters before their first meeting, she runs $2(100) = 200$ meters between their first and second meetings. Therefore the length of the track is $200 + 150 = 350$ meters.

16. **(B)** The given expression is defined if and only if

$$\log_{2003}(\log_{2002}(\log_{2001} x)) > 0,$$

that is, if and only if

$$\log_{2002}(\log_{2001} x) > 2003^0 = 1.$$

This inequality in turn is satisfied if and only if

$$\log_{2001} x > 2002,$$

that is, if and only if $x > 2001^{2002}$.

17. **(D)** Note that

$$\begin{aligned}
f(2^1) &= f(2) = f(2 \cdot 1) = 1 \cdot f(1) = 2^0 \cdot 2^0 = 2^0,\\
f(2^2) &= f(4) = f(2 \cdot 2) = 2 \cdot f(2) = 2^1 \cdot 2^0 = 2^1,\\
f(2^3) &= f(8) = f(2 \cdot 4) = 4 \cdot f(4) = 2^2 \cdot 2^1 \cdot 2^0 = 2^{(1+2)},\\
f(2^4) &= f(16) = f(2 \cdot 8) = 8 \cdot f(8) = 2^3 \cdot 2^2 \cdot 2^1 \cdot 2^0 = 2^{(1+2+3)},
\end{aligned}$$

and in general

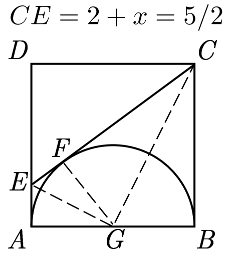
$$f(2^n) = 2^{(1+2+3+\ldots+(n-1))} = 2^{n(n-1)/2}$$

It follows that $f(2^{100}) = 2^{(100)(99)/2} = 2^{4950}$.

18. **(D)** Let F be the point at which $\overline{CE}$ is tangent to the semicircle, and let G be the midpoint of $\overline{AB}$. Because $\overline{CF}$ and $\overline{CB}$ are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = AE$. The Pythagorean Theorem applied to $\triangle CDE$ gives

$$(2 - x)^2 + 2^2 = (2 + x)^2.$$

It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



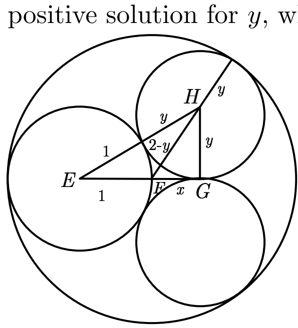
19. **(D)** Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$. Since the center of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2, which is the diameter of circle A. Applying the Pythagorean Theorem to right triangles EGH and FGH gives

$$(1+y)^2 = (1+x)^2 + y^2 \quad \text{and} \quad (2-y)^2 = x^2 + y^2,$$

from which it follows that

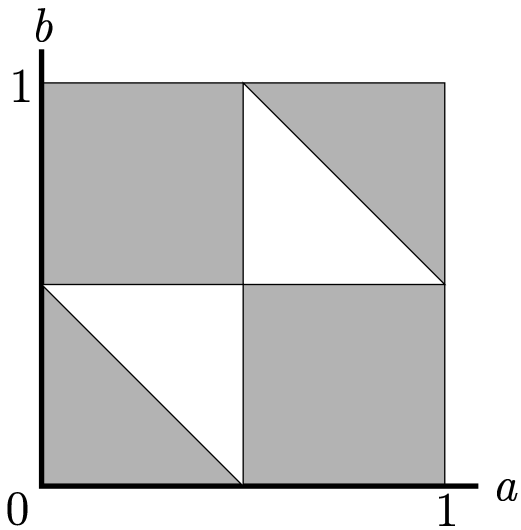
$$y = x + \frac{x^2}{2} \quad \text{and} \quad y = 1 - \frac{x^2}{4}.$$

The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (-2, 0)$. The radius of circle B is the positive solution for y, which is $8/9$.



20. **(E)** The conditions under which $A + B = C$ are as follows.
(i) If $a + b < 1/2$, then $A = B = C = 0$.
(ii) If $a \geq 1/2$ and $b < 1/2$, then $B = 0$ and $A = C = 1$.
(iii) If $a < 1/2$ and $b \geq 1/2$, then $A = 0$ and $B = C = 1$.
(iv) If $a + b \geq 3/2$, then $A = B = 1$ and $C = 2$.

These conditions correspond to the shaded regions of the graph shown. The combined area of those regions is $3/4$, and the area of the entire square is 1, so the requested probability is $3/4$.



21. **(D)** The given series is geometric with an initial term of 1 and a common ratio of $\cos^2\theta$, so its sum is

$$5 = \sum_{n=0}^{\infty} \cos^{2n}\theta = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Therefore $\sin^2\theta = \frac{1}{5}$, and

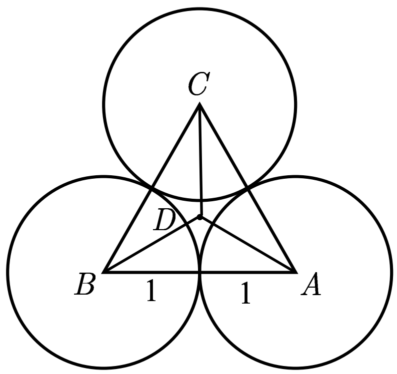
$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{5} = \frac{3}{5}.$$

22. **(B)** Let A, B, C and E be the centers of the three small spheres and the large sphere, respectively. Then $\triangle ABC$ is equilateral with side length 2. If D is the intersection of the medians of $\triangle ABC$, then E is directly above D. Because $AE = 3$ and $AD = 2\sqrt{3}/3$, it follows that

$$DE = \sqrt{3^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}.$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E, the distance from the plane to the top of the larger sphere is

$$3 + \frac{\sqrt{69}}{3}.$$



23. **(E)** Since $z_1 = 0$, it follows that $c_0 = P(0) = 0$. The nonreal zeros of P must occur in conjugate pairs, so $\sum_{k=1}^{2004} b_k = 0$ and $\sum_{k=1}^{2004} a_k = 0$ also. The coefficient c_{2003} is the sum of the zeros of P, which is

$$\sum_{k=1}^{2004} z_k = \sum_{k=1}^{2004} a_k + i \sum_{k=1}^{2004} b_k = 0.$$

Finally, since the degree of P is even, at least one of $z_2, \ldots, z_{2004}$ must be real, so at least one of $b_2, \ldots, b_{2004}$ is 0 and consequently $b_2 \cdot b_3 \cdot \ldots \cdot b_{2004} = 0$. Thus the quantities in **(A)**, **(B)**, **(C)**, and **(D)** must all be 0.

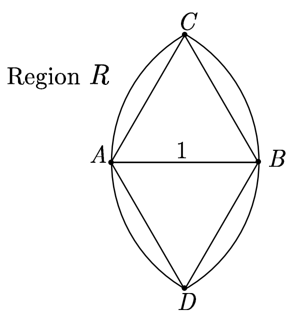
Note that the polynomial

$$P(x) = x(x-2)(x-3)\cdots(x-2003)\left(x + \sum_{k=2}^{2003} k\right)$$

satisfies the given conditions, and $\sum_{k=1}^{2004} c_k = P(1) \neq 0$.

24. **(C)** The center of the disk lies in a region R, consisting of all points within 1 unit of both A and B. Let C and D be the points of intersection of the circles of radius 1 centered at A and B. Because $\triangle ABC$ and $\triangle ABD$ are equilateral, arcs CAD and CBD are each 120°. Thus the sector bounded by $\overline{BC}$, $\overline{BD}$, and arc CAD has area $\pi/3$, as does the sector bounded by $\overline{AC}$, $\overline{AD}$, and arc CBD. The intersection of the two sectors, which is the union of the two triangles, has area $\sqrt{3}/2$, so the area of R is

$$\frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

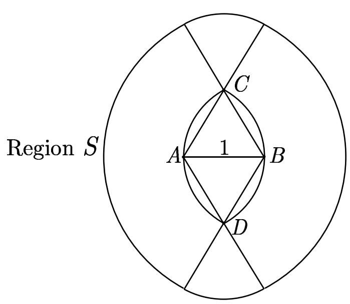


The region S consists of all points within 1 unit of R. In addition to R itself, S contains two $60°$ sectors of radius 1 and two $120°$ annuli of outer radius 2 and inner radius 1. The area of each sector is $\pi/6$, and the area of each annulus is

$$\frac{\pi}{3}(2^2 - 1^2) = \pi.$$

Therefore the area of S is

$$\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + 2\left(\frac{\pi}{6} + \pi\right) = 3\pi - \frac{\sqrt{3}}{2}.$$



25. **(E)** Note that $n^3 a_n = 133.\overline{133}_n = a_n + n^2 + 3n + 3$, so

$$a_n = \frac{n^2 + 3n + 3}{n^3 - 1} = \frac{(n+1)^3 - 1}{n(n^3 - 1)}.$$

Therefore

$$\begin{aligned} a_4 a_5 \cdots a_{99} &= \left(\frac{5^3 - 1}{4(4^3 - 1)}\right)\left(\frac{6^3 - 1}{5(5^3 - 1)}\right) \cdots \left(\frac{100^3 - 1}{99(99^3 - 1)}\right) \\ &= \left(\frac{3!}{99!}\right)\left(\frac{100^3 - 1}{4^3 - 1}\right) = \left(\frac{6}{99!}\right)\left(\frac{99(100^2 + 100 + 1)}{63}\right) \\ &= \frac{(2)(10{,}101)}{(21)(98!)} = \frac{962}{98!}. \end{aligned}$$

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