Tuesday, FEBRUARY 1, 2005

56th Annual American Mathematics Contest 12

AMC 12

Contest A

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 2.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 23rd annual American Invitational Mathematics Examination (AIME) on Tuesday, March 8, 2005 or Tuesday, March 22, 2005. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.

1. Two is 10% of x and 20% of y. What is $x - y$?

(A) 1 **(B)** 2 **(C)** 5 **(D)** 10 **(E)** 20

2. The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution for x. What is the value of b?

(A) -8 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 8

3. A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle?

(A) $\frac{1}{4}x^2$ **(B)** $\frac{2}{5}x^2$ **(C)** $\frac{1}{2}x^2$ **(D)** x^2 **(E)** $\frac{3}{2}x^2$

4. A store normally sells windows at $100 each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How many dollars will they save if they purchase the windows together rather than separately?

(A) 100 **(B)** 200 **(C)** 300 **(D)** 400 **(E)** 500

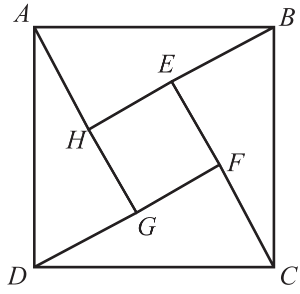
5. The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

(A) 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

6. Josh and Mike live 13 miles apart. Yesterday Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

7. Square $EFGH$ is inside square $ABCD$ so that each side of $EFGH$ can be extended to pass through a vertex of $ABCD$. Square $ABCD$ has side length $\sqrt{50}$, E is between B and H, and $BE = 1$. What is the area of the inner square $EFGH$?



(A) 25 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 42

8. Let $A, M,$ and C be digits with

$$(100A + 10M + C)(A + M + C) = 2005.$$

What is A?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

9. There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

(A) -16 **(B)** -8 **(C)** 0 **(D)** 8 **(E)** 20

10. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

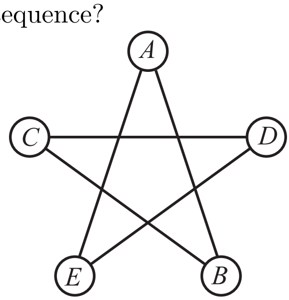
11. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

(A) 41 **(B)** 42 **(C)** 43 **(D)** 44 **(E)** 45

12. A line passes through $A(1, 1)$ and $B(100, 1000)$. How many other points with integer coordinates are on the line and strictly between A and B?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 8 **(E)** 9

13. In the five-sided star shown, the letters A, B, C, D and E are replaced by the numbers 3, 5, 6, 7 and 9, although not necessarily in that order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EA}$ form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?

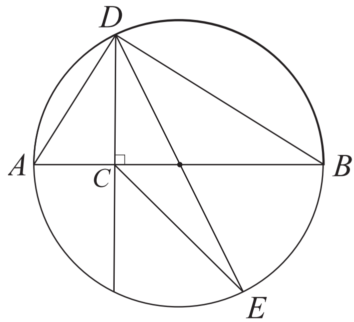


(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

14. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

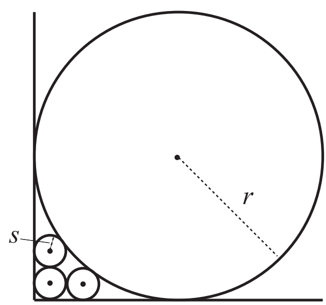
(A) $\frac{5}{11}$ **(B)** $\frac{10}{21}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{11}{21}$ **(E)** $\frac{6}{11}$

15. Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

16. Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



(A) 5 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 10

17. A unit cube is cut twice to form three triangular prisms, two of which are congruent, as shown in Figure 1. The cube is then cut in the same manner along the dashed lines shown in Figure 2. This creates nine pieces. What is the volume of the piece that contains vertex W?

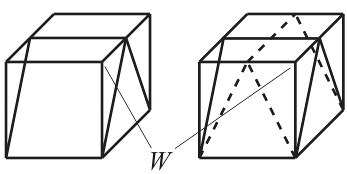


Figure 1 Figure 2

(A) $\frac{1}{12}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

18. Call a number "prime-looking" if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

(A) 100 **(B)** 102 **(C)** 104 **(D)** 106 **(E)** 108

19. A faulty car odometer proceeds from digit 3 to digit 5 , always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled?

(A) 1404 **(B)** 1462 **(C)** 1604 **(D)** 1605 **(E)** 1804

20. For each x in $[0, 1]$, define

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $f^{[2]}(x) = f(f(x))$, and $f^{[n+1]}(x) = f^{[n]}(f(x))$ for each integer $n \geq 2$. For how many values of x in $[0, 1]$ is $f^{[2005]}(x) = 1/2$?

(A) 0 **(B)** 2005 **(C)** 4010 **(D)** 2005^2 **(E)** 2^{2005}

21. How many ordered triples of integers (a, b, c), with $a \geq 2$, $b \geq 1$, and $c \geq 0$, satisfy both $\log_a b = c^{2005}$ and $a + b + c = 2005$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

22. A rectangular box P is inscribed in a sphere of radius r. The surface area of P is 384, and the sum of the lengths of its 12 edges is 112. What is r?

(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 16

23. Two distinct numbers a and b are chosen randomly from the set $\{2, 2^2, 2^3, \ldots, 2^{25}\}$. What is the probability that $\log_a b$ is an integer?

(A) $\frac{2}{25}$ **(B)** $\frac{31}{300}$ **(C)** $\frac{13}{100}$ **(D)** $\frac{7}{50}$ **(E)** $\frac{1}{2}$

24. Let $P(x) = (x - 1)(x - 2)(x - 3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) \cdot R(x)$?

(A) 19 **(B)** 22 **(C)** 24 **(D)** 27 **(E)** 32

25. Let S be the set of all points with coordinates (x, y, z), where x, y, and z are each chosen from the set $\{0, 1, 2\}$. How many equilateral triangles have all their vertices in S ?

(A) 72 **(B)** 76 **(C)** 80 **(D)** 84 **(E)** 88

WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

2005 AIME

The AIME will be held on Tuesday, March 8, 2005 with the alternate on March 22, 2005. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 19 and 20, 2005. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $10 (before shipping/handling fee), *PAYMENT IN U.S. FUNDS ONLY made payable to the* American Mathematics Competitions or VISA/MASTERCARD accepted. Include card number, expiration date, cardholder name, address, telephone and email. U.S.A. and Canadian orders must be prepaid and will be shipped Priority Mail, UPS or Air Mail.
INTERNATIONAL ORDERS: Do **NOT** prepay. An invoice will be sent to you.

Examinations: Each price is for one copy of an exam and its solutions. Specify the years you want and how many copies of each. All prices effective to September 1, 2005.

- AMC 10 2000–2005/(AHSME) AMC 12 1989–2005, $1 per exam copy.
- AIME 1983–1993, 1995–2005, $2 per copy per year (2005 available after March).
- USA and International Math Olympiads, 1989–1999, $5 per copy per year, (quantities limited)
- National Summary of Results and Awards, 1989–2005, $10 per copy per year.
- World Olympiad Problems/Solutions 1996–1997, 1997–1998, $15/ea.
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2005

AMC 12 – Contest A

DO NOT OPEN UNTIL TUESDAY, FEBRUARY 1, 2005

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 1.** Nothing is needed from inside this package until February 1.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

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