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SOLUTIONS PAMPHLET

Wednesday, March 22, 2006

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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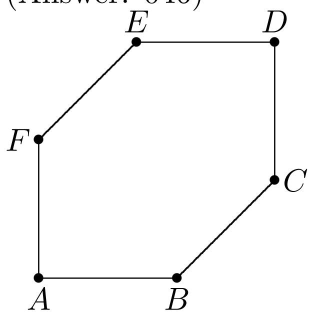
1. (Answer: 046)



Because $\angle B$, $\angle C$, $\angle E$, and $\angle F$ are congruent, the degree-measure of each of them is $\dfrac{720 - 2\cdot 90}{4} = 135$. Lines BF and CE divide the hexagonal region into two right triangles and a rectangle. Let $AB = x$. Then $BF = x\sqrt{2}$. Thus

$$2116(\sqrt{2}+1) = [ABCDEF] = 2\cdot \tfrac{1}{2}x^2 + x\cdot x\sqrt{2} = x^2(1+\sqrt{2}),$$

so $x^2 = 2116$, and $x = 46$.

2. (Answer: 893)

The Triangle Inequality yields

$$\log n < \log 75 + \log 12 = \log 900, \quad \text{and}$$
$$\log n > \log 75 - \log 12 = \log(25/4).$$

Therefore $25/4 < n < 900$, and so $7 \le n \le 899$. Hence there are $899 - 7 + 1 = 893$ possible values of n.

3. (Answer: 049)

Of the first 100 positive odd integers, $1, 3, 5, \ldots, 199$,

33 of them, namely $3, 9, 15, \ldots, 195 = 3(2\cdot 33 - 1)$, are divisible by 3;
11 of them, namely $9, 27, 45, \ldots, 189 = 9(2\cdot 11 - 1)$, are divisible by 9;
4 of them, namely $27, 81, 135, 189 = 27(2\cdot 4 - 1)$, are divisible by 27; and
1 of them, namely 81, is divisible by 81.

Therefore $k = 33 + 11 + 4 + 1 = 49$.

OR

Note that

$$P = \frac{200!}{2\cdot 4\cdot \cdots \cdot 200} = \frac{200!}{2^{100}\cdot 100!}.$$

The number of factors of 3 in the numerator is

$$\lfloor 200/3\rfloor + \lfloor 200/3^2\rfloor + \lfloor 200/3^3\rfloor + \lfloor 200/3^4\rfloor = 66 + 22 + 7 + 2 = 97,$$

and the number of factors of 3 in the denominator is

$$\lfloor 100/3\rfloor + \lfloor 100/3^2\rfloor + \lfloor 100/3^3\rfloor + \lfloor 100/3^4\rfloor = 33 + 11 + 3 + 1 = 48.$$

Therefore $k = 97 - 48 = 49$.

4. (Answer: 462)
Because a_6 is less than each of the other 11 numbers, $a_6 = 1$. Choose any five numbers of the remaining 11 to fill the first five positions. Their order is then uniquely determined. The order of the remaining six numbers which fill the last six positions is also uniquely determined. Thus the number of such permutations is the number of choices for the first five numbers, which is $\binom{11}{5} = 462$.

5. (Answer: 029)
Let $p(a, b)$ denote the probability of obtaining a on the first die and b on the second. Then the probability of obtaining a sum of 7 is

$$p(1,6) + p(2,5) + p(3,4) + p(4,3) + p(5,2) + p(6,1).$$

Let the probability of obtaining face F be $(1/6) + x$. Then the probability of obtaining the face opposite face F is $(1/6) - x$. Therefore

$$\begin{aligned}\frac{47}{288} &= 4\left(\frac{1}{6}\right)^2 + 2\left(\frac{1}{6} + x\right)\left(\frac{1}{6} - x\right) \\ &= \frac{4}{36} + 2\left(\frac{1}{36} - x^2\right) \\ &= \frac{1}{6} - 2x^2.\end{aligned}$$

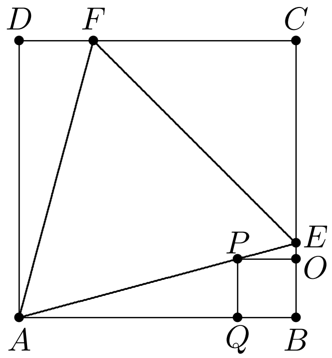
Then $2x^2 = 1/288$, and so $x = 1/24$. The probability of obtaining face F is therefore $(1/6) + (1/24) = 5/24$, and $m + n = 29$.

6. (Answer: 012)
Let $CF = x$. Then, because $\triangle ADF \cong \triangle ABE$, it follows that $DF = BE = 1 - x$, and $CE = x$. Hence $2x^2 = EF^2 = AE^2 = (1 - x)^2 + 1$, and so $x = \sqrt{3} - 1$. Let P and Q be the vertices of the smaller square that are on $\overline{AE}$ and $\overline{AB}$, respectively. Then

$$\begin{aligned}\frac{AB - PQ}{PQ} = \frac{AB - BQ}{PQ} = \frac{AQ}{PQ} &= \frac{AB}{BE}, \quad \text{so} \\ \frac{AB}{PQ} &= 1 + \frac{AB}{BE}, \quad \text{and} \\ \frac{1}{PQ} &= \frac{1}{AB} + \frac{1}{BE}.\end{aligned}$$

Thus $\dfrac{1}{PQ} = 1 + \dfrac{1}{1 - (\sqrt{3} - 1)} = 1 + \dfrac{1}{2 - \sqrt{3}} = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$. Consequently $PQ = \dfrac{1}{3 + \sqrt{3}} = \dfrac{3 - \sqrt{3}}{6}$, and $a + b + c = 12$.

OR



Let $BOPQ$ be the smaller square, where Q is between A and B, and let $BQ = y$. Then $QP = y$, and $AQ = y \tan 75°$. Thus $1 = AB = AQ + QB = y \tan 75° + y$, so $y = \dfrac{1}{1 + \tan 75°}$. But $\tan 75° = \tan(45° + 30°) = \dfrac{1 + (1/\sqrt{3})}{1 - (1/\sqrt{3})} = 2 + \sqrt{3}$. Therefore $y = \dfrac{1}{3 + \sqrt{3}} = \dfrac{3 - \sqrt{3}}{6}$.

OR

Place a coordinate system so that A is the origin, and the coordinates of B, C, and D are $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. Let $BE = p$. Then, as in the first solution, $p = 1 - (\sqrt{3} - 1) = 2 - \sqrt{3}$. Hence line AE has slope $2 - \sqrt{3}$ and contains the origin. Thus line AE has equation $y = (2 - \sqrt{3})x$. Let q be the length of a side of the smaller square. Then one vertex of that square has coordinates $(1 - q, q)$ and is on line AE. Therefore $q = (2 - \sqrt{3})(1 - q)$, which yields $q = (3 - \sqrt{3})/6$.

7. (Answer: 738)

Count the number of such ordered pairs with $a < b$. If a is a one-digit number, then b's digits are 9, 9, and $10 - a$. There are 9 choices for a in this case. In the case where a is a two-digit number, represent the digits of a as t and u. Then b's digits are 9, $9 - t$, and $10 - u$. Because $t \neq 0$ and $t \neq 9$, there are 8 choices for t and 9 choices for u, and so there are 72 choices for a. In the case where a is a three-digit number, represent the digits of a as h, t, and u. Then b's digits are $9 - h$, $9 - t$, and $10 - u$. Because $h = 1, 2, 3$, or 4, there are $4 \cdot 8 \cdot 9 = 288$ choices for a.

Thus the number of pairs with $a < b$ is $9 + 72 + 288 = 369$. Because each such pair can be reversed to give another allowable pair with $b < a$, there are $2 \cdot 369 = 738$ pairs.

OR

Count the number of forbidden pairs, that is, pairs in which a or b has a zero digit. If a or b has units digit 0, then both do, and the given equation reduces to $r + s = 100$, where $a = 10r$ and $b = 10s$. Thus, in this case, there are 99 forbidden pairs.

In the case where neither a nor b has units digit 0, then exactly one of them must be of the form $h0u$, where neither h nor u is 0. There are $9 \cdot 9 = 81$ such values of a and 81 such values for b for a total of 162 forbidden pairs in this case. Therefore the total number of forbidden pairs is $99 + 162 = 261$, and there are $999 - 261 = 738$ of the requested pairs.

8. (Answer: 336)
Because any permutation of the vertices of a large triangle can be obtained by rotation or reflection, the coloring of the large triangle is determined by which set of three colors is used for the corner triangles and the color that is used for the center triangle. If the three corner triangles are the same color, there are six possible sets of colors for them. If exactly two of the corner triangles are the same color, there are $6 \cdot 5 = 30$ possible sets of colors. If the three corner triangles are different colors, there are $\binom{6}{3} = 20$ possible sets of colors. Therefore there are $6 + 30 + 20 = 56$ sets of colors for the corner triangles. Because there are six choices for the color of the center triangle, there are $6 \cdot 56 = 336$ distinguishable triangles.

9. (Answer: 027)
Let O_1, O_2, and O_3 be the centers of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively, let A and B be the points where t_1 is tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and let D and E be the points where t_2 is tangent to $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. Radii $\overline{O_1A}$ and $\overline{O_2B}$ are perpendicular to line AB. Let P be the intersection of $\overline{AB}$ and $\overline{O_1O_2}$. Then $\triangle O_1AP \sim \triangle O_2BP$ with similarity ratio $1:2$. Therefore $O_1P = 4$ and $O_2P = 8$, so $PB = \sqrt{8^2 - 2^2} = 2\sqrt{15}$. The slope of line t_1 is equal to $\tan \angle BPO_2 = 1/\sqrt{15}$, so line t_1 has equation $y = (1/\sqrt{15})(x - 4)$. Similarly, let Q be the intersection of $\overline{DE}$ and $\overline{O_2O_3}$, and conclude that $O_2Q = 4$ and $O_3Q = 8$, and then that $DQ = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. The slope of line t_2 is equal to $\tan \angle DQO_3 = -\tan \angle DQO_2 = -1/\sqrt{3}$, so line t_2 has equation $y = (-1/\sqrt{3})(x - 16)$. Now $(1/\sqrt{15})(x - 4) = (-1/\sqrt{3})(x - 16)$ implies $x - 4 = -\sqrt{5}(x - 16)$, so $x = \dfrac{16\sqrt{5} + 4}{\sqrt{5} + 1} = 19 - 3\sqrt{5}$, and $p + q + r = 27$.

10. (Answer: 831)
Each team has to play six games in all. Team A and team B each has 5 more games to play, and they do not play against each other, for a total of $2^5 \cdot 2^5$ possible outcomes. For team A to finish with more points, it has to win at least as many games as team B does. The number of outcomes in which the two teams win the same number of games is

$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2 = 252.$$

Of the remaining $1024 - 252 = 772$ outcomes, A wins more than B in half of them. Thus the requested probability is $\dfrac{252 + (772/2)}{1024} = \dfrac{319}{512}$, and $m + n = 831$.

OR

Team A and team B each has 5 more games to play. For $1 \le k \le 5$, the kth game for team A and the kth game for team B will change the difference between the scores of team A and team B by $+1$, 0, or -1, and a change of 0 is twice as likely as each of the other changes. Hence consider the coefficients of the generating function $g(x) = (x^{-1} + 2 + x)^5$, and find the sum of all the coefficients of terms of the form x^k, where $k \ge 0$. Note that

$$g(x) = \frac{(1 + 2x + x^2)^5}{x^5} = \frac{(1 + x)^{10}}{x^5}.$$

Thus the sum is

$$\binom{10}{5} + \binom{10}{6} + \cdots + \binom{10}{10} = \frac{1}{2}\left(\sum_{i=0}^{10}\binom{10}{i} + \binom{10}{5}\right) = \frac{1}{2}\left(2^{10} + \binom{10}{5}\right) = 638,$$

so the requested probability is $638/4^5 = 319/512$, and $m + n = 831$.

11. (Answer: 834)
Because $a_{k+3} - a_{k+2} = a_{k+1} + a_k$ for all positive integers k, conclude that

$$\sum_{k=1}^{n}(a_{k+3} - a_{k+2}) = \sum_{k=1}^{n}(a_{k+1} + a_k).$$

Let $S_n = \sum_{k=1}^{n} a_k$. Notice that $\sum_{k=1}^{n}(a_{k+3} - a_{k+2})$ telescopes to $a_{n+3} - a_3$, and that $\sum_{k=1}^{n}(a_{k+1} + a_k) = (S_n - a_1 + a_{n+1}) + S_n$. Therefore $a_{n+3} - a_3 = S_n - a_1 + a_{n+1} + S_n$, so $S_n = (1/2)(a_{n+3} - a_{n+1}) = (1/2)(a_{n+2} + a_n)$, and in particular $S_{28} = (1/2)(a_{30} + a_{28})$. Thus the last three digits of the sum are the same as those of $(1/2)(3361 + 0307)$, namely 834, and the requested remainder is 834.

12. (Answer: 865)
Notice that $\angle GCB \cong \angle GAB$ and $\angle CAG \cong \angle CBG$ because each pair of angles intercepts the same arc. Also $\angle CAG \cong \angle AFD$ because $\overline{AE} \parallel \overline{DF}$. Thus $\triangle AFD \sim \triangle CBG$, and $[CBG] = t^2[AFD]$, where t is the similarity ratio. Because $m\angle ADF = 120^\circ$, $[AFD] = (1/2)AD \cdot DF \cdot \sin 120^\circ = 143\sqrt{3}/4$. The length of each side of $\triangle ABC$ is $2\sqrt{3}$. The Law of Cosines implies that $AF^2 = 13^2 + 11^2 - 2 \cdot 13 \cdot 11(-1/2) = 433$, so $AF = \sqrt{433}$. Therefore $t = \dfrac{BC}{AF} = \dfrac{2\sqrt{3}}{\sqrt{433}}$, so

$$[CBG] = t^2 \cdot [AFD] = \left(\frac{2\sqrt{3}}{\sqrt{433}}\right)^2 \cdot \frac{143\sqrt{3}}{4} = \frac{429\sqrt{3}}{433},$$

and $p + q + r = 865$.

OR

Let $\alpha = m\angle DAF$, and let $\beta = m\angle EAF$. Then $m\angle BCG = \alpha$ and $m\angle CBG = \beta$. Note that $[BGC] = (1/2)BC \cdot CG \sin\alpha$, and apply the Law of Sines in $\triangle BGC$ to conclude that $\dfrac{BC}{\sin 120^\circ} = \dfrac{CG}{\sin\beta}$. Then $[BGC] = \dfrac{1}{2} \cdot BC \cdot \dfrac{BC\sin\beta}{\sin 120^\circ}\sin\alpha = \dfrac{BC^2\sin\alpha\sin\beta}{\sqrt{3}}$. Use the Law of Cosines in $\triangle AEF$ to conclude that $AF = \sqrt{433}$, and use the Law of Sines to conclude that $\dfrac{\sqrt{433}}{\sin 120^\circ} = \dfrac{13}{\sin\beta}$, so $\sin\beta = \dfrac{13\sqrt{3}}{2\sqrt{433}}$. The Law of Sines implies that $\sin\alpha = \sin\angle AFE = \dfrac{11\sqrt{3}}{2\sqrt{433}}$. Thus

$$[BGC] = \frac{\left(2\sqrt{3}\right)^2\left(\frac{11\sqrt{3}}{2\sqrt{433}}\right)\left(\frac{13\sqrt{3}}{2\sqrt{433}}\right)}{\sqrt{3}} = \frac{429\sqrt{3}}{433}.$$

13. (Answer: 015)

Recall that the sum of the first m positive odd integers is m^2. Thus if N is equal to the sum of the $(k+1)$th through mth positive odd integers, then $N = m^2 - k^2 = (m-k)(m+k)$. Let $a = m-k$, and let $b = m+k$. Note that $a \le b$, and a and b have the same parity. Thus N is either odd or a multiple of 4. Conversely, if $N = ab$, where a and b are positive integers with the same parity and $a \le b$, then $N = m^2 - k^2$, where $m = (b+a)/2$ and $k = (b-a)/2$, and it follows that N is the sum of the $(k+1)$th through mth odd integers. Thus there is a one-to-one correspondence between the sets of consecutive positive odd integers whose sum is N and the ordered pairs (a,b) of positive integers such that a and b are of the same parity, $ab = N$, and $a \le b$.

First consider the case where N is odd. All the divisors of N have the same parity because they are all odd. Since five pairs of positive integers have product N, N must have either 9 or 10 divisors. N must therefore have the form p^8, p^9, p^2q^2, or pq^4, where p and q are distinct odd primes. But N cannot have the form p^8 or p^9, because that would imply that $N \ge 3^8 > 1000$. If N has the form p^2q^2, then $pq \le 31$ because $N < 1000$, and there are two possible values of N, namely $3^2 \cdot 5^2$ and $3^2 \cdot 7^2$. If N has the form pq^4, then N must be $5 \cdot 3^4$, $7 \cdot 3^4$, or $11 \cdot 3^4$.

In the case where N is even, $N = ab$, where $a = 2a'$ and $b = 2b'$ for positive integers a' and b'. In this case, N has five pairs of divisors of the same parity if and only if $N/4$ has 9 or 10 divisors. Count the number of positive integers less than 250 that are of the previously mentioned forms, except that now p or q can be 2. There are no integers less than 250 that are of the form p^8 or p^9; there are four such integers of the form p^2q^2, namely, $2^2 \cdot 3^2$, $2^2 \cdot 5^2$, $2^2 \cdot 7^2$, and $3^2 \cdot 5^2$; and there are six such integers of the form pq^4, namely, $3 \cdot 2^4$, $5 \cdot 2^4$, $7 \cdot 2^4$, $11 \cdot 2^4$, $13 \cdot 2^4$, and $2 \cdot 3^4$.

Thus there are a total of $2 + 3 + 4 + 6 = 15$ possible values for N.

14. (Answer: 063)

Each of the 10^n integers from 0 to $10^n - 1$, inclusive, can be written as an n-digit string, using leading 0's as necessary. Imagine these strings written one beneath the other to form a table of digits with n columns and 10^n rows. Each column contains an equal number of digits of each type, so there are $(1/10) \cdot 10^n$ digits of each type in each column, and there are $(n/10) \cdot 10^n = n \cdot 10^{n-1}$ digits of each type in the table. Therefore

$$S_n = 1 + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) n \cdot 10^{n-1}.$$

The sum S_n is not an integer when $n = 1, 2$, or 3, and when $n \geq 4$,

$$\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8}\right) n \cdot 10^{n-1} \quad \text{and} \quad \left(\frac{1}{3} + \frac{1}{6}\right) n \cdot 10^{n-1} = \frac{1}{2} n \cdot 10^{n-1}$$

are integers. Thus S_n is an integer when

$$\left(\frac{1}{7} + \frac{1}{9}\right) n \cdot 10^{n-1} = \frac{16n}{63} \cdot 10^{n-1}$$

is an integer. Because $16 \cdot 10^{n-1}$ and 63 are relatively prime, the smallest value of n for which S_n is an integer is 63.

15. (Answer: 009)

The radicals on the right side of the first equation are reminiscent of the Pythagorean Theorem. Each radical represents the length of a leg of a right triangle whose other leg has length 1/4, and whose hypotenuse has length y or z. Adjoin these two triangles along the leg of length 1/4 to create $\triangle XYZ$ with $x = YZ$, $y = ZX$, and $z = XY$, and with altitude to side $\overline{YZ}$ of length 1/4. Because of similar considerations in the other two equations, let the lengths of the altitudes to sides $\overline{XZ}$ and $\overline{XY}$ be 1/5 and 1/6, respectively. In the $\triangle XYZ$ thus created, the lengths x, y, and z of the sides satisfy the given equations, provided the altitudes of $\triangle XYZ$ are inside the triangle, that is, provided $\triangle XYZ$ is acute.

In general, a triangle the lengths of whose sides are a, b, and c is acute if and only if $a^2 + b^2 > c^2$, where $a \leq b \leq c$. Denote the area of the triangle by K and the lengths of the altitudes to the sides of lengths a, b, and c by h_a, h_b, and h_c, respectively. Then $K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$, so the condition $a^2 + b^2 > c^2$ is equivalent to $(1/h_a)^2 + (1/h_b)^2 > (1/h_c)^2$, where $1/h_a \leq 1/h_b \leq 1/h_c$. Thus $\triangle XYZ$ is acute because $4^2 + 5^2 > 6^2$.

Let K be the area of $\triangle XYZ$. Then $x = 8K$, $y = 10K$, and $z = 12K$, so $x + y + z = 30K$. Apply Heron's Formula to obtain $K^2 = 15K \cdot 7K \cdot 5K \cdot 3K$. Because $K > 0$, conclude that $K = 1/(15\sqrt{7})$. Then $x + y + z = 30K = 2/\sqrt{7}$, so $m + n = 9$.

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