Mathematical Association of America

American Mathematics Competitions

28th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME I)

Tuesday, March 16, 2010

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO & the USAJMO will be given in your school on TUESDAY and WEDNESDAY, April 27 & 28, 2010.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2. Find the remainder when

$$9 \cdot 99 \cdot 999 \cdot \cdots \cdot \underbrace{99\ldots9}_{\text{999 9's}}$$

is divided by 1000.

3. Suppose that $y = \frac{3}{4}x$ and $x^y = y^x$. The quantity $x+y$ can be expressed as a rational number $\frac{r}{s}$, where r and s are relatively prime positive integers. Find $r+s$.

4. Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m+n$.

5. Positive integers a, b, c, and d satisfy $a > b > c > d$, $a+b+c+d = 2010$, and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of a.

6. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \le P(x) \le 2x^2 - 4x + 3$$

for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

7. Define an ordered triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of sets to be *minimally intersecting* if $|\mathcal{A} \cap \mathcal{B}| = |\mathcal{B} \cap \mathcal{C}| = |\mathcal{C} \cap \mathcal{A}| = 1$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = \emptyset$. For example, $(\{1,2\}, \{2,3\}, \{1,3,4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when N is divided by 1000.

   Note: $|\mathcal{S}|$ represents the number of elements in the set $\mathcal{S}$.

8. For a real number a, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25.$$

The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\frac{\sqrt{m}}{n}$, where m and n are integers and m is not divisible by the square of any prime. Find $m+n$.

9. Let (a, b, c) be a real solution of the system of equations

$$\begin{aligned} x^3 - xyz &= 2 \\ y^3 - xyz &= 6 \\ z^3 - xyz &= 20. \end{aligned}$$

The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10. Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

11. Let $\mathcal{R}$ be the region consisting of the set of points in the coordinate plane that satisfy both $|8 - x| + y \le 10$ and $3y - x \ge 15$. When $\mathcal{R}$ is revolved around the line whose equation is $3y - x = 15$, the volume of the resulting solid is $\dfrac{m\pi}{n\sqrt{p}}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.

12. Let $m \ge 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a, b, and c (not necessarily distinct) such that $ab = c$.

Note: a partition of S is a pair of sets A, B such that $A \cap B = \emptyset$, $A \cup B = S$.

13. Rectangle $ABCD$ and a semicircle with diameter $\overline{AB}$ are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment $\overline{AB}$, and segment $\overline{CD}$ at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1:2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

14. For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \le 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

15. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

Your Exam Manager will receive a copy of the 2010 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2010 USA(J)MO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) is a 6-question, 9-hour, essay-type examination. The USA(J)MO will be held in your school on Tuesday and Wednesday, April 27 & 28, 2010. Your teacher has more details on who qualifies for the USA(J)MO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC -- www.unl.edu/amc/d-publication/publication.shtml
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