MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

28th Annual *(alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME II)

Wednesday, March 31, 2010

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO and the USAJMO will be given in your school on TUESDAY and WEDNESDAY, April 27 and 28, 2010.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Let N be the greatest integer multiple of 36 all of whose digits are even and no two of whose digits are the same. Find the remainder when N is divided by 1000.

2. A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. The probability that $\frac{1}{5} \le d(P) \le \frac{1}{3}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3. Let K be the product of all factors $(b-a)$ (not necessarily distinct) where a and b are integers satisfying $1 \le a < b \le 20$. Find the greatest positive integer n such that 2^n divides K.

4. Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

6. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

7. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i, w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

8. Let N be the number of ordered pairs of nonempty sets $\mathcal{A}$ and $\mathcal{B}$ that have the following properties:

   - $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
   - $\mathcal{A} \cap \mathcal{B} = \emptyset$,
   - The number of elements of $\mathcal{A}$ is not an element of $\mathcal{A}$,
   - The number of elements of $\mathcal{B}$ is not an element of $\mathcal{B}$.

   Find N.

9. Let $ABCDEF$ be a regular hexagon. Let G, H, I, J, K, and L be the midpoints of sides AB, BC, CD, DE, EF, and AF, respectively. The segments AH, BI, CJ, DK, EL, and FG bound a smaller regular hexagon. Let the ratio of the area of the smaller hexagon to the area of $ABCDEF$ be expressed as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

10. Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$.

11. Define a *T-grid* to be a 3×3 matrix which satisfies the following two properties:

    (1) Exactly five of the entries are 1's, and the remaining four entries are 0's,

    (2) Among the eight rows, columns, and long diagonals (the long diagonals are $\{a_{13}, a_{22}, a_{31}\}$ and $\{a_{11}, a_{22}, a_{33}\}$), no more than one of the eight has all three entries equal.

    Find the number of distinct T-grids.

12. Two noncongruent integer-sided isosceles triangles have the same perimeter and the same area. The ratio of the lengths of the bases of the two triangles is $8 : 7$. Find the minimum possible value of their common perimeter.

13. The 52 cards in a deck are numbered $1, 2, \ldots, 52$. Alex, Blair, Corey, and Dylan each picks a card from the deck without replacement and with each card being equally likely to be picked. The two persons with lower numbered cards form a team, and the two persons with higher numbered cards form another team. Let $p(a)$ be the probability that Alex and Dylan are on the same team, given that Alex picks one of the cards a and $a + 9$, and Dylan picks the other of these two cards. The minimum value of $p(a)$ for which $p(a) \geq \frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. In right triangle ABC with the right angle at C, $\angle BAC < 45°$ and $AB = 4$. Point P on $\overline{AB}$ has the properties that $\angle APC = 2\angle ACP$ and $CP = 1$. The ratio $\dfrac{AP}{BP}$ can be represented in the form $p + q\sqrt{r}$, where p, q and r are positive integers and r is not divisible by the square of any prime. Find $p + q + r$.

15. In triangle ABC, $AC = 13$, $BC = 14$, and $AB = 15$. Points M and D lie on $\overline{AC}$ with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on $\overline{AB}$ with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the other point of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets $\overline{BC}$ at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.

Your Exam Manager will receive a copy of the 2010 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2010 USA(J)MO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) are 6-question, 9-hour, essay-type examinations. The USA(J)MO will be held in your school on Tuesday and Wednesday, April 27 and 28, 2010. Your teacher has more details on who qualifies for the USA(J)MO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC -- www.unl.edu/amc/d-publication/publication.shtml
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