Mathematical Association of America
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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME II)

SOLUTIONS PAMPHLET

Wednesday, March 31, 2010

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 640)

   Because $36 = 4 \cdot 9$ and $\gcd(4, 9) = 1$, it suffices to verify divisibility by 4 and by 9 separately. Because 9 divides N, the sum of the digits of N is divisible by 9. Because the digits of N are even, their sum must be divisible by 18, and hence the set of possible digits of N is $\{0, 4, 6, 8\}$. The maximum value of N formed by these digits is 8640, which is divisible by 4. Thus $N = 8640$, and the requested remainder is 640.

2. (Answer: 281)

   Note that because the square has area 1, the requested probability is equal to the area of the region determined by the given conditions. For $0 < r < 1$, let S_r denote the square concentric with S which has side length r. Every point P inside S except its center lies on the boundary of S_r for exactly one r, and for such a point, the distance $d(P)$ is $\frac{1-r}{2}$. The given inequality is satisfied if P is inside $S_{3/5}$ but outside $S_{1/3}$. This occurs with probability
   $$\frac{9}{25} - \frac{1}{9} = \frac{56}{225},$$
   and the requested sum is 281.

3. (Answer: 150)

   The product K contains nineteen 1's $(2-1, 3-2, 4-3, \ldots, 20-19)$, eighteen 2's $(3-1, 4-2, 5-3, \ldots, 20-18)$, and so forth. Thus $K = 1^{19} \cdot 2^{18} \cdot 3^{17} \cdot 4^{16} \cdots 19^1$. The power of 2 in this product is $2^{18} \cdot 4^{16} \cdot 2^{14} \cdot 8^{12} \cdot 2^{10} \cdot 4^8 \cdot 2^6 \cdot 16^4 \cdot 2^2$. The number of factors of 2 is therefore
   $$1 \cdot 18 + 2 \cdot 16 + 1 \cdot 14 + 3 \cdot 12 + 1 \cdot 10 + 2 \cdot 8 + 1 \cdot 6 + 4 \cdot 4 + 1 \cdot 2 = 150.$$

4. (Answer: 052)

   Let D_i be the event that the original departure gate was i, and N_i be the event that the new gate is i. Then
   $$\begin{aligned}
   &P(\text{distance} \le 400 \text{ ft}) \\
   &= \sum_{i=1}^{4} P(D_i)P(N_1 \text{ through } N_{i+4}) + \sum_{i=5}^{8} P(D_i)P(N_{i-4} \text{ through } N_{i+4}) \\
   &\quad + \sum_{i=9}^{12} P(D_i)P(N_{i-4} \text{ through } N_{12}) \\
   &= 2 \cdot \frac{1}{12}\left(\frac{4}{11} + \frac{5}{11} + \frac{6}{11} + \frac{7}{11}\right) + \frac{1}{12}\left(4 \cdot \frac{8}{11}\right) \\
   &= \frac{11}{33} + \frac{8}{33} = \frac{19}{33},
   \end{aligned}$$

and $m+n=52$.

5. (Answer: 075)

Let $a=\log_{10} x$, $b=\log_{10} y$, and $c=\log_{10} z$. Take the log of each side of the equation $xyz=10^{81}$ to obtain $\log_{10} xyz = a+b+c=81$. Now square each side of this equation to obtain $a^2+b^2+c^2+2ab+2ac+2bc=81^2$. Note that $(\log_{10} x)(\log_{10} yz)=(\log_{10} x)(\log_{10} y+\log_{10} z)=ab+ac$. Thus $(\log_{10} x)^2+(\log_{10} y)^2+(\log_{10} z)^2=81^2-2\cdot 468=5625$, and the answer is $\sqrt{5625}=75$.

Note: There are an infinite number of values of a, b, and c which satisfy the conditions of the problem, including $a=69, b=6+6\sqrt{11}$, and $c=6-6\sqrt{11}$.

6. (Answer: 008)

If $ax-b$ is a factor of the given polynomial, then $a=1$ and b is a root. Thus $n=b^3+\frac{63}{b}$, which achieves a minimum integer value of 48 when $b=3$.

On the other hand, suppose

$$x^4-nx+63=(ax^2+bx+c)(dx^2+ex+f),$$

where all coefficients are integers. By multiplying both factors by -1 if necessary, it can be assumed that $a>0$; thus $ad=1$ implies $a=d=1$. Equating coefficients for x^3 implies that $b+e=0$, so

$$\begin{aligned} x^4-nx+63 &= x^4-(c+f-b^2)x^2-(bc-bf)x+cf \\ &= x^4+(c+f-b^2)x^2+b(f-c)x+cf. \end{aligned}$$

The coefficient of x^2 is $c+f-b^2$, and the constant term is $cf=63$. Thus $c+f=b^2$, and so c and f are positive. The pairs of positive factors of 63 sum to $1+63=64$, $3+21=24$, $7+9=16$, of which only the first and the last are squares. In the first case, $b=\pm 8$, and

$$(x^2\pm 8x+63)(x^2\mp 8x+1)=x^4\mp 496x+63.$$

In the second case, $b=\pm 4$, and

$$(x^2\pm 4x+9)(x^2\mp 4x+7)=x^4\mp 8x+63.$$

Thus the smallest possible value of n in this case is 8, which is less than the value in the previous case and hence the minimum.

7. (Answer: 136)

Let $w=x+yi$, where x and y are real. Then because a is real and the sum of the three roots is $-a$, it follows that $\text{Im}((w+3i)+(w+9i)+(2w-4))=0$.

Thus $y+3+y+9+2y=0$, and $y=-3$. Therefore the three roots are $x, x+6i$, and $2x-4-6i$. Because the coefficients of $P(z)$ are real, the non-real roots must occur in conjugate pairs, and so $x=2x-4$ and $x=4$. Thus $P(z)=(z-4)(z-(4+6i))(z-(4-6i))$ and $1+a+b+c=P(1)=(-3)(-3-6i)(-3+6i)=-135$. Thus $|a+b+c|=|-135-1|=136$. Such a polynomial exists: $P(z)=z^3-12z^2+84z-208$ has the zeros $4, 4\pm 6i$, which satisfy the conditions of the problem for $w=4-9i$.

8. (Answer: 772)

Let $|\mathcal{M}|$ represent the number of elements in the set $\mathcal{M}$.

Let $|\mathcal{A}|=k$. Then the first two properties imply that $|\mathcal{B}|=12-k$, and because $\mathcal{A}$ and $\mathcal{B}$ are nonempty, it follows that $k\neq 0$ and $k\neq 12$. The last two properties imply that $k\notin\mathcal{A}$ and $12-k\notin\mathcal{B}$. Thus the first property implies that $k\in\mathcal{B}$ and $12-k\in\mathcal{A}$. Furthermore, k cannot equal 6, because otherwise, $|\mathcal{A}|=|\mathcal{B}|=6$. Thus $6\in\mathcal{A}\cap\mathcal{B}$, which violates the second property. After assigning k to $\mathcal{B}$ and $12-k$ to $\mathcal{A}$, the remaining $k-1$ elements of $\mathcal{A}$ can be chosen in $\binom{10}{k-1}$ ways, and the remaining $11-k$ elements must belong to set $\mathcal{B}$. Thus

$$N=\left(\sum_{k=1}^{11}\binom{10}{k-1}\right)-\binom{10}{6-1}=2^{10}-252=772,$$

and the answer is 772.

9. (Answer: 011)

Without loss of generality, let $AB=2$, and place $ABCDEF$ in the first and second quadrants of the coordinate plane with $A=(0,0)$ and $B=(2,0)$. Then $C=(3,\sqrt{3})$, $E=(0,2\sqrt{3})$, $F=(-1,\sqrt{3})$, $G=(1,0)$, $H=\left(\frac{5}{2},\frac{\sqrt{3}}{2}\right)$, and $L=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Then line AH has equation $y=\frac{\sqrt{3}}{5}x$, line FG has equation $y=\frac{-\sqrt{3}}{2}x+\frac{\sqrt{3}}{2}$, and line EL has equation $y=(3\sqrt{3})x+2\sqrt{3}$. The intersection of lines AH and FG is then $X=\left(\frac{5}{7},\frac{\sqrt{3}}{7}\right)$, and the intersection of lines EL and FG is $Y=\left(\frac{-3}{7},\frac{5\sqrt{3}}{7}\right)$. Then $\overline{XY}$ is a side of the smaller hexagon, and the ratio of the areas is the square of the ratio of the sides, which is

$$\left(\frac{XY}{2}\right)^2=\left(\frac{1}{2}\sqrt{\left(\frac{8}{7}\right)^2+\left(-\frac{4\sqrt{3}}{7}\right)^2}\right)^2=\frac{\frac{64}{49}+\frac{48}{49}}{4}=\frac{112}{196}=\frac{4}{7},$$

so $m+n=11$.

10. (Answer: 163)

If $f(x) = c(x - r_1)(x - r_2)$, the product cr_1r_2 must be $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. For $1 \le k \le 4$, if c has $4 - k$ prime factors, there are $\binom{4}{k}$ choices for the k prime factors of 2010 that divide r_1r_2. Of these, there are 2^k choices for the factors dividing r_1; the others must divide r_2. The roots of each polynomial obtained in this way are distinct and each possible pair of roots is counted exactly twice. Therefore there are $\binom{4}{k} \cdot 2^{k-1}$ choices for the two roots, up to sign. Furthermore, an even number of $\{c, r_1, r_2\}$ must be negative. This gives $\binom{3}{0} + \binom{3}{2} = 4$ possible assignments of signs for each of the $\sum_{k=1}^{4} \binom{4}{k} \cdot 2^{k-1} = 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 4 + 1 \cdot 8 = 40$ choices of $\{|c|, |r_1|, |r_2|\}$. Finally, if $|c| = 2010$ then $|r_1| = |r_2| = 1$. There are only three sign assignments that give rise to distinct polynomials in this case, because both cases in which $r_1 = -r_2$ give rise to the same polynomial. Combining this with the prior discussion, there are $4 \cdot 40 + 3 = 163$ such polynomials in all.

11. (Answer: 068)

Note that choosing 5 of the 8 entries to be 1's shows that the number of 3×3 matrices satisfying (1) is $\binom{9}{5}$ or 126. Counting the number of 3×3 matrices satisfying (1) but not (2) and subtracting from 126 will then produce the required answer.

The 3×3 matrices satisfying (1) but not (2) fall into two groups: those 3×3 matrices which have one row, column, or long diagonal of all 1's and one row, column, or long diagonal of all 0's, and those 3×3 matrices which have two rows, columns, or diagonals which are both all 1's or both all 0's. In the first group note that no matrix can have both a row of 1's and a column or diagonal of 0's, both a column of 1's and a row or diagonal of 0's, or both a diagonal of 1's and a row or column of 0's. Thus the only members of the first group are matrices with one row of 1's and one row of 0's or with one column of 1's and one column of 0's. There are 6 ways to choose the row or column of 1's, 2 ways to choose the row or column of 0's, and then 3 ways to fill in the remaining three entries (that is, 110, 101, or 011). Thus the first group has $6 \cdot 2 \cdot 3$ or 36 members. The members of the second group must, by the previous argument, have two overlapping groups (rows, columns, or diagonals) of the same number. But two overlapping groups of 0's would require $3 + 3 - 1 = 5$ 0's, and the matrix only contains four 0's. Thus the second group consists of those 3×3 matrices with both a row and a column of 1's, both a row or column and a diagonal of 1's, or two diagonals of 1's. Because the two groups of 1's require five 1's, the remaining entries in the matrix must all be 0's. There are $3 \cdot 3 = 9$ matrices of the first type, $6 \cdot 2 = 12$ matrices of the second type, and 1 matrix of the third type. Thus the total number of matrices in the second group is $9 + 12 + 1 = 22$ matrices. The number of matrices satisfying (1) but not (2) is therefore 36+22 or 58. The requested number of T-grids is then $126 - 58 = 68$.

12. (Answer: 338)

Let the length of the base of one of the triangles be $8a$ and let the length of the base of the other triangle be $7a$, for some positive integer a. Because these two triangles have the same area, the lengths of the corresponding altitudes must be $7h$ and $8h$. Because the perimeters of the triangles are equal, it follows that

$$8a + 2\sqrt{16a^2 + 49h^2} = 7a + 2\sqrt{\frac{49}{4}a^2 + 64h^2}, \text{ or}$$
$$a + 2\sqrt{16a^2 + 49h^2} = 2\sqrt{\frac{49}{4}a^2 + 64h^2}.$$

Squaring both sides of the last equation and simplifying gives

$$a^2 + 64a^2 + 196h^2 + 4a\sqrt{16a^2 + 49h^2} = 49a^2 + 256h^2, \text{ or}$$
$$a\sqrt{16a^2 + 49h^2} = 15h^2 - 4a^2.$$

Squaring both sides of this equation and simplifying yields

$$a^2(16a^2 + 49h^2) = 225h^4 - 120a^2h^2 + 16a^4, \text{ or } 225h^2 = 169a^2.$$

Thus $h = \frac{13a}{15}$, and the common perimeter is

$$8a + 2\sqrt{16a^2 + 49h^2} = 8a + \frac{218a}{15}.$$

Because the perimeter p is an increasing function of a, it must attain its minimum for the smallest acceptable value of a. The triangle are integer-sided, and therefore the value of p must also be an integer. Because 218 and 15 have no common factors, the smallest value of a for which p is an integer is 15. Thus the value requested is

$$8 \cdot 15 + \frac{436 \cdot 15}{15} = 120 + 218 = 338.$$

13. (Answer: 263)

Alex and Dylan are on the same team if Blair and Corey picked cards numbered b and c with either $1 \le b,\ c \le a - 1$ or $a + 10 \le b,\ c \le 52$ from the 50 cards from the deck excluding the cards numbered a and $a + 9$. Thus

$$p(a) = \frac{(a-1)(a-2) + (43-a)(42-a)}{50 \cdot 49} = \frac{a^2 - 44a + 904}{25 \cdot 49}.$$

Because $p(a) \geq \frac{1}{2}$, it follows that

$$p(a) = \frac{a^2 - 44a + 904}{25 \cdot 49} \geq \frac{1}{2},$$

and thus

$$(a-22)^2 + 420 \geq \frac{25 \cdot 49}{2}.$$

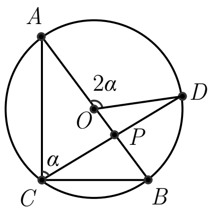
Hence $(a-22)^2 \geq \frac{385}{2}$. Because a is an integer, it follows that $a - 22 \geq 14$ or $a - 22 \leq -14$; that is, $a \geq 36$ or $a \leq 8$. Thus the minimum possible value of $p(a)$ is equal to

$$p(8) = p(36) = \frac{88}{175},$$

and the requested sum is 263.

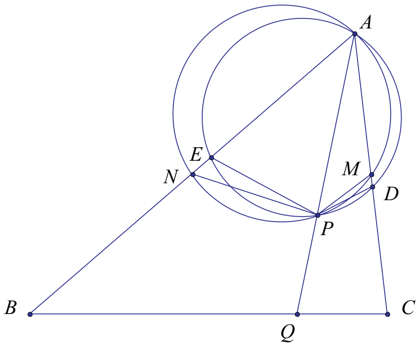
14. (Answer: 007)

Let the circumcircle of $\triangle ABC$ have center at O and radius r, and let $\angle ACP = \alpha$. Extend $\overline{CP}$ to intersect the circle at the point D. Because $\angle AOD = \angle DPB = 2\alpha$, it follows that $DO = DP = r$. Because inscribed angles subtended by the same arc are equal, it follows that $\triangle APD$ and $\triangle CPB$ are similar. Therefore $\dfrac{CP}{BP} = \dfrac{AP}{DP}$ and $\dfrac{CP}{AP} = \dfrac{BP}{DP}$. Thus $\dfrac{CP}{BP} + \dfrac{CP}{AP} = \dfrac{AP}{DP} + \dfrac{BP}{DP} = \dfrac{AB}{DP} = \dfrac{2r}{r} = 2$. Observe that $\angle BAC < 45^\circ$ implies that $AP > BP$. Because $CP = 1$, the previous equation takes the form $\dfrac{1}{4 - AP} + \dfrac{1}{AP} = 2$, giving $2 + \sqrt{2} = AP$. It follows that $BP = 2 - \sqrt{2}$, and so $\dfrac{AP}{BP} = \dfrac{2+\sqrt{2}}{2-\sqrt{2}} = 3 + 2\sqrt{2}$. Hence $p + q + r = 7$.



Note: The existence of such a triangle can be shown by using Stewart's Theorem.

15. (Answer: 218)



The Angle Bisector Theorem implies that E lies on $\overline{AN}$ and D lies on $\overline{MC}$ because $AE/EB = AC/BC < 1$ and $AD/DC = AB/CB > 1$. The Angle Bisector Theorem furthermore implies

$$NE = AN - AE = \frac{AB}{2} - \frac{AC}{AC+BC} \cdot AB = \frac{5}{18}$$

and

$$MD = CM - CD = \frac{AC}{2} - \frac{BC}{BC+BA} \cdot AC = \frac{13}{58}.$$

Because $ANPM$ is cyclic, $\angle ENP = \angle ANP = \angle PMD$. Because $AEPD$ is cyclic, $\angle NEP = 180° - \angle AEP = \angle ADP = \angle MDP$. Because $\angle ENP = \angle PMD$ and $\angle NEP = \angle MDP$, triangles NEP and MDP are similar. Hence

$$\frac{NE}{MD} = \frac{NP}{MP}.$$

Applying the Law of Sines to $\triangle ANP$ and $\triangle AMP$ gives

$$\frac{NE}{MD} = \frac{NP}{MP} = \frac{\sin\angle NAP}{\sin\angle PAM} = \frac{\sin\angle BAQ}{\sin\angle QAC}$$

and thus

$$\frac{\sin\angle BAQ}{\sin\angle QAC} = \frac{\left(\frac{5}{18}\right)}{\left(\frac{13}{58}\right)} = \frac{145}{117}.$$

Thus

$$\frac{BQ}{QC} = \frac{\text{Area}(ABQ)}{\text{Area}(ACQ)} = \frac{\frac{1}{2}\cdot AB\cdot AQ\cdot\sin\angle BAQ}{\frac{1}{2}\cdot AC\cdot AQ\cdot\sin\angle QAC} = \frac{15}{13}\cdot\frac{145}{117} = \frac{725}{507},$$

and $m - n = 218$.