AP® Calculus AB
2003 Scoring Guidelines

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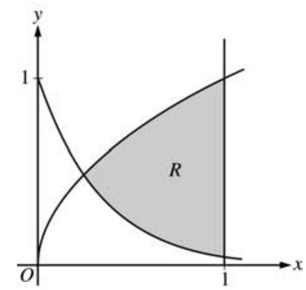
Question 1

Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 1$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

Point of intersection

$e^{-3x} = \sqrt{x}$ at $(T, S) = (0.238734, 0.488604)$

1: Correct limits in an integral in (a), (b), or (c)

(a) $\text{Area} = \int_T^1 \left(\sqrt{x} - e^{-3x}\right) dx = 0.442$ or 0.443

2: { 1 : integrand; 1 : answer }

(b) $\text{Volume} = \pi \int_T^1 \left(\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2\right) dx = 0.453\pi$ or 1.423 or 1.424

3: {
2 : integrand
  < −1 > reversal
  < −1 > error with constant
  < −1 > omits 1 in one radius
  < −2 > other errors
1 : answer
}

(c) $\text{Length} = \sqrt{x} - e^{-3x}$

$\text{Height} = 5\left(\sqrt{x} - e^{-3x}\right)$

$\text{Volume} = \int_T^1 5\left(\sqrt{x} - e^{-3x}\right)^2 dx = 1.554$

3: {
2 : integrand
  < −1 > incorrect but has $\sqrt{x} - e^{-3x}$ as a factor
1 : answer
}

Question 2

A particle moves along the x-axis so that its velocity at time t is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

At time $t = 0,$ the particle is at position $x = 1.$

(a) Find the acceleration of the particle at time $t = 2.$ Is the speed of the particle increasing at $t = 2$? Why or why not?

(b) Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3.$

(d) During the time interval $0 \le t \le 3,$ what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

(a) $a(2) = v'(2) = 1.587$ or 1.588 $v(2) = -3\sin(2) < 0$ Speed is decreasing since $a(2) > 0$ and $v(2) < 0$.	2 : { 1 : $a(2)$ 1 : speed decreasing with reason
(b) $v(t) = 0$ when $\frac{t^2}{2} = \pi$ $t = \sqrt{2\pi}$ or 2.506 or 2.507 Since $v(t) < 0$ for $0 < t < \sqrt{2\pi}$ and $v(t) > 0$ for $\sqrt{2\pi} < t < 3$, the particle changes directions at $t = \sqrt{2\pi}$.	2 : { 1 : $t = \sqrt{2\pi}$ only 1 : justification
(c) Distance $= \int_0^3 \lvert v(t) \rvert\, dt = 4.333$ or 4.334	3 : { 1 : limits 1 : integrand 1 : answer
(d) $\int_0^{\sqrt{2\pi}} v(t)\, dt = -3.265$ $x(\sqrt{2\pi}) = x(0) + \int_0^{\sqrt{2\pi}} v(t)\, dt = -2.265$ Since the total distance from $t = 0$ to $t = 3$ is 4.334, the particle is still to the left of the origin at $t = 3.$ Hence the greatest distance from the origin is 2.265.	2 : { 1 : ± (distance particle travels while velocity is negative) 1 : answer

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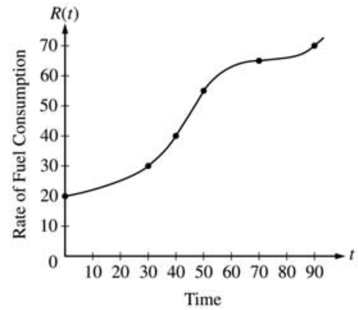
Question 3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

(a) $R'(45) \approx \dfrac{R(50) - R(40)}{50 - 40} = \dfrac{55 - 40}{10}$
$= 1.5 \text{ gal/min}^2$

2 :
- 1 : a difference quotient using numbers from table and interval that contains 45
- 1 : 1.5 gal/min^2

(b) $R''(45) = 0$ since $R'(t)$ has a maximum at $t = 45$.

2 :
- 1 : $R''(45) = 0$
- 1 : reason

(c) $\int_0^{90} R(t)\,dt \approx (30)(20) + (10)(30) + (10)(40)$
$+ (20)(55) + (20)(65) = 3700$

Yes, this approximation is less because the graph of R is increasing on the interval.

2 :
- 1 : value of left Riemann sum
- 1 : "less" with reason

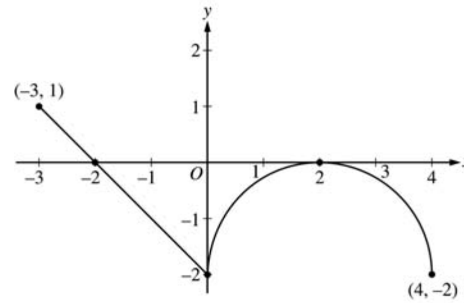
(d) $\int_0^{b} R(t)\,dt$ is the total amount of fuel in gallons consumed for the first b minutes.

$\frac{1}{b}\int_0^{b} R(t)\,dt$ is the average value of the rate of fuel consumption in gallons/min during the first b minutes.

3 :
- 2 : meanings
  - 1 : meaning of $\int_0^{b} R(t)\,dt$
  - 1 : meaning of $\frac{1}{b}\int_0^{b} R(t)\,dt$
  - $< -1 >$ if no reference to time b
- 1 : units in both answers

Question 4

Let f be a function defined on the closed interval $-3 \le x \le 4$ with $f(0) = 3$. The graph of f', the derivative of f, consists of one line segment and a semicircle, as shown above.



Graph of f'

(a) On what intervals, if any, is f increasing? Justify your answer.

(b) Find the x-coordinate of each point of inflection of the graph of f on the open interval $-3 < x < 4$. Justify your answer.

(c) Find an equation for the line tangent to the graph of f at the point $(0, 3)$.

(d) Find $f(-3)$ and $f(4)$. Show the work that leads to your answers.

(a) The function f is increasing on $[-3, -2]$ since $f' > 0$ for $-3 \le x < -2$.

$2: \begin{cases} 1: \text{interval} \\ 1: \text{reason} \end{cases}$

(b) $x = 0$ and $x = 2$

f' changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 2$

$2: \begin{cases} 1: x = 0 \text{ and } x = 2 \text{ only} \\ 1: \text{justification} \end{cases}$

(c) $f'(0) = -2$

Tangent line is $y = -2x + 3$.

1 : equation

(d) $$f(0) - f(-3) = \int_{-3}^{0} f'(t)\,dt = \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = -\frac{3}{2}$$

$$f(-3) = f(0) + \frac{3}{2} = \frac{9}{2}$$

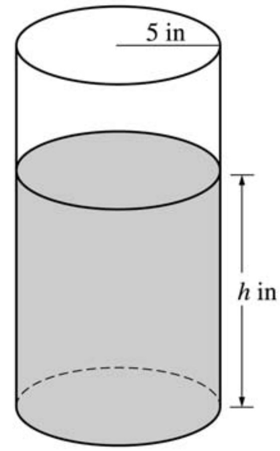
$$f(4) - f(0) = \int_{0}^{4} f'(t)\,dt = -\left(8 - \frac{1}{2}(2)^2\pi\right) = -8 + 2\pi$$

$$f(4) = f(0) - 8 + 2\pi = -5 + 2\pi$$

4 :
- 1 : $\pm\left(\frac{1}{2} - 2\right)$ (difference of areas of triangles)
- 1 : answer for $f(-3)$ using FTC
- 1 : $\pm\left(8 - \frac{1}{2}(2)^2\pi\right)$ (area of rectangle – area of semicircle)
- 1 : answer for $f(4)$ using FTC

Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

3 : { 1 : $\dfrac{dV}{dt} = -5\pi\sqrt{h}$; 1 : computes $\dfrac{dV}{dt}$; 1 : shows result }

(b) $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

5 : { 1 : separates variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition $h = 17$ when $t = 0$; 1 : solves for h }

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\dfrac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

Question 6

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

(a) f is continuous at $x = 3$ because

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = 2.$

Therefore, $\lim\limits_{x\to 3} f(x) = 2 = f(3).$

$2 : \begin{cases} 1 : \text{answers “yes” and equates the values of the left- and right-hand limits} \\ 1 : \text{explanation involving limits} \end{cases}$

(b) $$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= \frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{1}{2}x^2\right)\Big|_3^5$$

$$= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(\frac{25}{2} - \frac{21}{2}\right) = \frac{20}{3}$$

Average value: $\frac{1}{5}\int_0^5 f(x)\,dx = \frac{4}{3}$

$4 : \begin{cases} 1 : k\int_0^3 f(x)\,dx + k\int_3^5 f(x)\,dx \\ \quad (\text{where } k \ne 0) \\ 1 : \text{antiderivative of } \sqrt{x+1} \\ 1 : \text{antiderivative of } 5 - x \\ 1 : \text{evaluation and answer} \end{cases}$

(c) Since g is continuous at $x = 3$, $2k = 3m + 2$.

$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & \text{for } 0 < x < 3 \\ m & \text{for } 3 < x < 5 \end{cases}$$

$\lim\limits_{x\to 3^-} g'(x) = \frac{k}{4}$ and $\lim\limits_{x\to 3^+} g'(x) = m$

Since these two limits exist and g is differentiable at $x = 3$, the two limits are equal. Thus $\frac{k}{4} = m$.

$8m = 3m + 2$; $m = \frac{2}{5}$ and $k = \frac{8}{5}$

$3 : \begin{cases} 1 : 2k = 3m + 2 \\ 1 : \frac{k}{4} = m \\ 1 : \text{values for } k \text{ and } m \end{cases}$