2019 AP Calculus AB Free Response Questions: Tentative Solutions

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Note: All of these solutions represent my work on the free response problems and the points awarded are my best guess as to how the college board will award points. I have included a thorough explanation of each solution to help the students understand the solutions.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

Solution 1. Below is the tentative solution for FRQ 1.

(a) To find the amount of fish that entered the lake in the 5 hour period, we need to integrate $E(t)$ from $t = 0$ to $t = 5$.

$$\int_0^5 E(t)dt = 153.458$$

Since we were asked to round to the nearest whole number, the answer is 153
2 ***points:*** 1 ***point for the integral and*** 1 ***point for the answer.***

(b) The average number of fish that leave the lake is given by

$$\frac{1}{5}\int_0^5 L(t)dt = 6.059$$

2 ***points:*** 1 ***point for the integral expression and*** 1 ***point for the answer.***

(c) Suppose $A(t)$ is the amount of fish in the lake at time t. Then we know $A'(t) = E(t) - L(t)$. To find the absolute maximum of $A(t)$ on $[0, 8]$, we need evaluate $A(0)$, $A(8)$ and $A(c)$ at any critical point c. To find critical points, we solve $A'(t) = E(t) - L(t) = 0$ which occurs at $t \approx 6.204$. Therefore, $c \approx 6.204$. Since we weren't given $A(0)$, let us see what the Fundamental Theorem of Calculus I tells us about the relationship between $A(c)$ and $A(0)$.

$$A(c) - A(0) = \int_0^c E(t) - L(t)dt \approx 135.015$$

. Therefore $A(c) \approx A(0) + 135.015 > A(0)$.
We can do the same to find the relationship between $A(8)$ and $A(c)$,

$$A(8) - A(c) = \int_c^8 E(t) - L(t)dt \approx -54.095$$

Therefore, $A(8) \approx A(c) - 54.095 < A(c)$.
This means that $t = c$ is when the absolute max occurs at $t = c$.
3 ***points:*** 1 ***point for considering*** $E(t) - L(t) = 0$***,*** 1 ***point for finding*** $t = 6.204$***, and*** 1 ***point for the answer with justification.***

(d) To see if the rate of change in the number of fish is decreasing or increasing we need to check the derivative of $A'(t)$ at $t = 5$. In other words, we will evaluate $A''(t) = E'(5) - L'(5) \approx -10.723 < 0$. Since $A''(5)$ is negative, then the rate of change of the number of fish is decreasing at $t = 5$.
2 ***points:*** 1 ***point for considering*** $E'(5) - L'(5)$ ***and*** 1 ***point for the answer with an explanation.***

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	−29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\, dt$.

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

Solution 2. Below is the tentative solution for FRQ 2.

(a) Since v_P is said to be a differentiable function, it is differentiable on all real numbers. Therefore, it is differentiable over the open interval $(0.3, 2.8)$. Since differentiable functions are continuous, then we know that v_P is continuous on the closed interval $[0.3, 2.8]$. Therefore, the conditions for the Mean Value Theorem are satisfied, which means that we know there exists a $c \in (0.3, 2.8)$ such that the acceleration $v_P'(c) = \dfrac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = \dfrac{0}{2.5} = 0$.
2 ***points:*** 1 ***point for explaining that*** $v_P(t)$ ***satisfies conditions of Mean Value Theorem,*** 1 ***point for correct different quotient.***

(b) Using a trapezoidal sum, we get

$$\frac{55+0}{2}(0.3) + \frac{-29+55}{2}(1.7-0.3) + \frac{55+-29}{2}(2.8-1.7) \approx \int_0^{2.8} v_P(t)dt$$

I hope no one was silly enough to actually simplify this, but if you were

$$\int_0^{2.8} v_P(t)dt \approx 40.75$$

2 ***points:*** 1 ***point for trapezoidal sum,*** 1 ***point for answer.***

(c) To find when particle Q is traveling, at least, 60 meters per hour, we need to find when $v_Q(t) \ge 60$. The easiest way to see this is to graph $y = v_Q(t)$ and $y = 60$ and see where they intersect. Once we've graphed both functions, we can see that $v_Q(t) \ge 60$ for all t, such that $1.867 \le t \le 3.519$.
The second part of the questions asks that we find the distance traveled by particle Q during this time. To compute the distance traveled, we use

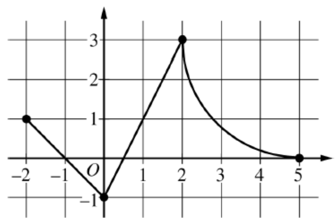
$$\int_{1.867}^{3.519} |v_Q(t)|dt \approx 106.049 \text{ meters}$$

2 ***points:*** 1 ***point for interval on which*** $v_Q(t) \ge 60$ ***,*** 1 ***point for distance traveled with integral expression.***

(d) We know that particle P starts at the origin. In part (b), we estimated that the integral $\int_0^{2.8} v_p(t)dt \approx$ 102.55. Therefore, the position of the particle, P, $x_P(2.8) \approx 102.55$.
To find the position of particle Q, we use the Fundamental Theorem of Calculus I.

$$\begin{aligned} x_Q(2.8) - x_Q(0) &= \int_0^{2.8} v_Q(t)dt \\ x_Q(2.8) - -90 &= 135.938 \\ x_Q(2.8) &= -90 + 135.938 \\ x_Q(2.8) &= 45.938 \end{aligned}$$

Therefore, the distance between particles Q and P is $|45.938 - 40.75| \approx 5.118$
3 ***points:*** 1 ***point for position of particle*** P ***at*** $t = 2.8$***,*** 1 ***point for position of particle*** Q ***at*** $t = 2.8$***,*** 1 ***point for the distance between the particles.***



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Solution 3. Below is the tentative solution for FRQ 3.

(a) Integration properties tell us that

$$\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{5} f(x)dx$$

Since we know the left side of the equation is 7 and we can compute the right side of the equation using that the integral of f from -2 to 5 is the signed area between the graph of f and the x-axis, we can use the equation above to solve for the desired integral.The problem we can foresee is that the line from the point $(0,-1)$ to the point $(3,3)$ crosses the x-axis at an unknown point. We can find this point using the equation of that line, but we will do that separately. For now, we will consider the signed area between f and x-axis from $x = 0$ to $x = 3$, $A = \int_0^3 f(x)dx$.

$$\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{5} f(x)dx$$
$$\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{3} f(x)dx + \int_{3}^{5} f(x)dx$$
$$7 = \int_{-6}^{-2} f(x)dx + [(\frac{1}{2}(1)(1) + \frac{1}{2}(-1)(1) + A + (9 - \frac{1}{4}(9\pi))]$$
$$7 = \int_{-6}^{-2} f(x)dx + A + [9 - \frac{9\pi}{4}]$$
$$\frac{9\pi}{4} - 2 - A = \int_{-6}^{-2} f(x)dx$$

To find the area A, we can use two different methods:
Method 1:
The slope of the line from $(0,-1)$ to $(3,3)$ is 2, so the equation of that line is $y = 2x - 1$. To find where

the line crosses the x-axis, we need to set $y = 0$, yielding, $x = \frac{1}{2}$. Therefore, the area A is given by

$$A = \frac{1}{2}(-1)(\frac{1}{2}) + \frac{1}{2}(3)(\frac{3}{2}) = 2$$

Method 2:
We can consider this a trapezoid with bases of signed length -1 and 3 (bases are parallel to each other) and a height of 2 (the distance from $x = 0$ to $x = 2$), and use the trapezoidal formula for area:

$$A = \frac{-1+3}{2}(2) = 2$$

Therefore, since

$$\frac{9\pi}{4} - 2 - A = \int_{-6}^{-2} f(x)dx$$

Then, substituting $A = 2$, we get

$$\frac{9\pi}{4} - 4 = \int_{-6}^{-2} f(x)dx$$

2 ***points:*** 1 ***point for the integration property and*** 1 ***points for the answer.***

(b) To evaluate

$$\int_{3}^{5} (2f'(x) + 4)dx$$

we need to use integration properties and the Fundamental Theorem of Calculus.

$$\begin{aligned}
\int_{3}^{5} (2f'(x) + 4)dx &= 2\int_{3}^{5} f'(x)dx + \int_{3}^{5} 4dx \\
&= 2[f(5) - f(3)] + 4x\Big|_{3}^{5} \\
&= 2f(5) - 2f(3) + 4(5-3) \\
&= 2(0) - 2f(3) + 8 \\
&= 8 - 2f(3)
\end{aligned}$$

We still need to find $f(3)$. We know that $f(3)$ is the y-coordinate of the point on the circle corresponding to $x = 3$. The equation of the circle of radius 3 centered at $(5, 3)$ is given by

$$(x-5)^2 + (y-3)^2 = 9$$

We will plug in $x = 3$ and solve for y.

$$\begin{aligned}
(3-5)^2 + (y-3)^2 &= 9 \\
4 + (y-3)^2 &= 9 \\
(y-3)^2 &= 5 \\
y - 3 &= \pm\sqrt{5} \\
y &= 3 \pm \sqrt{5}
\end{aligned}$$

We can see that the y coordinate is less than 3, so $y \neq 3+\sqrt{5}$, which means that $y = 3-\sqrt{5}$. Therefore,

$$\int_{3}^{5} (2f'(x) + 4)dx = 8 - 2(3 - \sqrt{5}) = 2 + 2\sqrt{5}$$

3 ***points:*** 1 ***point for using the Fundamental Theorem of Calculus,*** 1 ***point for using integration properties,*** 1 ***point for answer.***

(c) In order to find the absolute maximum of $g(x)$, we need to evaluate $g(x)$ at the endpoints and at the relative maxima. By the Fundamental Theorem of Calculus II, we know that $g'(x) = f(x)$. By analyzing when the graph of f intersects the x-axis, we can see that the critical points occur at $x = -1, \frac{1}{2}$, and 5. However, since $g' = f$ only changes from positive to negative at $x = -1$, we need only evaluate $g(x)$ at $x = -1$.

$$g(-1) = \int_{-2}^{-1} f(t)dt = \frac{1}{2}$$

Now we need to evaluate $g(x)$ at the endpoints.

$$g(-2) = \int_{-2}^{-2} f(t)dt = 0$$

and

$$g(5) = \int_{-2}^{5} f(t)dt = 11 - \frac{9\pi}{4}$$

Since

$$3 < \pi < 4$$

then

$$27 < 9\pi < 36$$

so

$$6 < \frac{27}{4} < \frac{9\pi}{4} < 9$$

Multiplying by -1, we get

$$-6 > \frac{-9\pi}{4} > -9$$

which means

$$5 > 11 - 6 > 11 - \frac{9\pi}{4} > 11 - 9 = 2$$

In particular, $11 - \frac{9\pi}{4} > \frac{1}{2} > 0$, so it is the absolute maximum value of $g(x)$ on the interval $[-2, 5]$.
3 ***points:*** 1 ***point for considering*** $x = -1$***,*** 1 ***point for analysis of*** g***,*** 1 ***point for the answer with justification.***

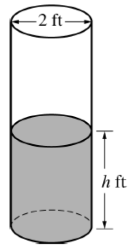
(d) This limit can be computed directly if we realize that 10^x and $\arctan(x)$ are continuous everywhere. Also, since f is a line around $x = 1$, then it is infinitely differentiable, which means f and f' are also continuous at $x = 1$. Therefore, to compute the limit, we only need to evaluate $\frac{10^x - 3f'(x)}{f(x) - \arctan(x)}$ at $x = 1$.
Since $f'(1)$ is the slope of the graph of $f(x)$ at $x = 1$, and $f(1)$ is the y-value of the graph of f at $x = 1$, then we know that $f'(1) = 2$ and $f(1) = 1$. Therefore,

$$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan(x)} = \frac{10^1 - 3(2)}{1 - \arctan(1)}$$

There is no need to simplify, but if you insist on simplifying, you should get

$$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan(x)} = \frac{4}{1 - \frac{\pi}{4}} = \frac{4}{\frac{4-\pi}{4}} = \frac{16}{4 - \pi}$$

2 ***points:*** 1 ***point for value of*** $f(1)$ ***or*** $f'(1)$***,*** 1 ***point for answer.***



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

Solution 4. Below is the tentative solution for FRQ 4.

(a) The rate of change of the volume with respect to time is the derivative of $V = \pi r^2 h$ with respect to t. Notice, that since the container is in the shape of a cylinder, the radius is constant, which means that we can use $r = 1$ as the radius from the beginning, $V = \pi h$.

$$\begin{aligned} V &= \pi h \\ \frac{dV}{dt} &= \pi \frac{dh}{dt} \\ \frac{dV}{dt} &= \pi \frac{-1}{10}\sqrt{h} \end{aligned}$$

Since we are asked to find the rate of change of the volume when $h = 4$, then

$$\frac{dV}{dt} = \pi \frac{-1}{10}\sqrt{4} = -\frac{\pi}{5} \text{ cubic feet per second}$$

2 ***points:*** 1 ***point for*** $\frac{dV}{dt}$***,*** 1 ***point for answer with units.***

(b) To see if the rate of change of the height is increasing or decreasing, we need to take the derivative of $\frac{dh}{dt}$ with respect to t.

$$\begin{aligned} \frac{d}{dt}\left(\frac{dh}{dt}\right) &= \frac{d}{dt}\left(-\frac{1}{10}\sqrt{h}\right) \\ &= \frac{d}{dt}\left(-\frac{1}{10}h^{\frac{1}{2}}\right) \\ &= -\frac{1}{10}\left(\frac{1}{2}h^{-\frac{1}{2}}\frac{dh}{dt}\right) \\ &= -\frac{1}{20\sqrt{h}}\frac{-1}{10}\sqrt{h} \\ &= \frac{1}{200} \end{aligned}$$

Since $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for all t, then the rate of change of the height is increasing. 2 ***points:*** 1 ***point for derivative of*** $\dfrac{dh}{dt}$***,*** 1 ***point for answer with reasoning.***

(c) To solve this differential equation we need to separate the variables:

$$\frac{dh}{\sqrt{h}} = -\frac{1}{10}dt$$

Integrate both sides:

$$\int h^{-\frac{1}{2}}\, dh = \int -\frac{1}{10}dt$$

but don't forget the C:

$$2h^{\frac{1}{2}} = -\frac{1}{10}t + C$$

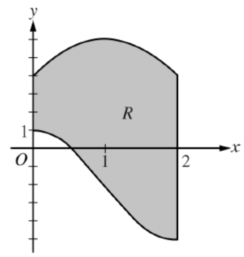
Now, we plug in the initial condition:

$$2\sqrt{5} = -\frac{1}{10}(0) + C$$

resulting in $C = 2\sqrt{5}$. Now we can solve for h:

$$\begin{aligned} 2\sqrt{h} &= -\frac{1}{10}t + 2\sqrt{5} \\ \sqrt{h} &= -\frac{1}{20}t + \sqrt{5} \\ h &= (\sqrt{5} - \frac{1}{20}t)^2 \end{aligned}$$

where $t \geq 0$.
5 ***points:*** 1 ***point for separation of variables,*** 1 ***point for integrating both sides,*** 1 ***point for constant of integration,*** 1 ***point for initial condition,*** 1 ***point for*** h***.***



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

Solution 5. Below is the tentative solution for FRQ 5.

(a) To find the area of R, we, first, need to identify which graph is h and which graph is g.
Since $h(x) = 6 - 2(x-1)^2$, then we know that at $x = 1$, $h(1) = 6$, so the point $(1, 6)$ must be on the graph of h.
Therefore, we can see that $h(x) > g(x)$ for all $x \in [0, 2]$, so it is the graph on "top".

$$
\begin{aligned}
\text{Area } &= \int_0^2 (h(x) - g(x))\,dx \\
&= \int_0^2 \left(6 - 2(x-1)^2 - (-2 + 3\cos(\frac{\pi}{2}x))\right) dx \\
&= \int_0^2 6 - 2(x-1)^2 + 2 - 3\cos(\frac{\pi}{2}x)dx \\
&= 6x - \frac{2}{3}(x-1)^3 + 2x - 3\sin(\frac{\pi}{2}x)\frac{1}{\frac{\pi}{2}}\Big|_0^2 \\
&= 8x - \frac{2}{3}(x-1)^3 - \frac{6}{\pi}\sin(\frac{\pi}{2}x)\Big|_0^2 \\
&= \left(8(2) - \frac{2}{3}(2-1)^3 - \frac{6}{\pi}\sin(\frac{\pi}{2}(2))\right) - (8(0) - \frac{2}{3}(0-1)^3 - \frac{6}{\pi}\sin(\frac{\pi}{2}(0))) \\
&= 16 - \frac{2}{3} - \frac{2}{3} \\
&= \frac{44}{3}
\end{aligned}
$$

Recall, that you may leave the answer as it is in the third to last line and still get full credit.
3 ***points:*** 1 ***point for integral,*** 1 ***point for antiderivatives,*** 1 ***point for answer.***

(b) Since the volume of the solid is the integral of area of the cross-section, and they gave us the area function for the cross-section, then the volume of the solid is

$$\begin{aligned}\text{Volume} &= \int_0^2 A(x)dx \\ &= \int_0^2 \frac{1}{x+3}dx \\ &= \ln|x+3|\Big|_0^2 \\ &= \ln(5) - \ln(3) \\ &= \ln(\frac{5}{3})\end{aligned}$$

3 ***points:*** 1 ***point for integral,*** 1 ***point for anti-derivative,*** 1 ***point for answer.***

(c) If the region R is rotated around the line $y = 6$, then the cross-sections perpendicular to the axis of rotation would be "washers". The area of these washers are given by $A(x) = \pi R^2(x) - \pi r^2(x)$ where $R(x)$ is the radius function for the big circle, and $r(x)$ is the radius function of the smaller circle. The radius of the larger circle is the distance from $y = 6$ to the graph of the function furthest from $y = 6$, which happens to be $g(x)$, so $R(x) = 6 - g(x)$. The radius of the smaller circle is the distance from the axis of rotation, $y = 6$ and the graph of the function closer to it, $h(x)$. Therefore, the integral expression for the volume of the solid is given by

$$\int_0^2 \pi(6 - g(x)^2 - \pi(6 - h(x))^2 dx$$

2 ***points:*** 2 ***points for integrand***

1 ***point for limits of integration in parts (a), (b), and (c).***

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

Solution 6. Below is the tentative solution for FRQ 6.

(a) We can find $h'(2)$ by interpreting $h'(2)$ as the slope of the tangent line at $x = 2$. Since we are told that the line given by the equation $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both $h(x)$ and $g(x)$ at $x = 2$, then we know that the slope of the tangent to $h(x)$ at $x = 2$ is $\frac{2}{3}$.

$h'(2) = \frac{2}{3}$.

1 ***point for answer***

(b) To find $a'(2)$ we must first find $a'(x)$, which we need to compute using product rule.

$$a'(x) = 9x^2 h(x) + 3x^2 h'(x)$$

Now we can substitute $x = 2$, to get

$$a'(2) = 9(4)h(2) + 3(8)h'(2) = 36(4) + 24(\frac{2}{3}) = 120 + 24 + 16 = 160$$

2 ***points:*** 1 ***point for derivative,*** 1 ***point for answer***

(() c) Since we know that h is twice-differentiable, then we know that h is continuous. This means that

$$4 = h(2) = \lim_{x \to 2} h(x)$$

Therefore, we know that the limit

$$\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$$

Since we are told that we can use L'Hôpital's rule, we know that $\lim_{x \to 2} 1 - (f(x))^3 = 0$.
We also know that f is also twice-differentiable, then it is also continuous, so $1 - (f(2))^3 = 0$ which

means that $1 = f(2)$. Now we will derive another equation by using L'Hôpital's rule on $\lim_{x\to 2} h(x)$.

$$\begin{aligned} 4 &= \lim_{x\to 2} h(x) \\ 4 &= \lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} \\ 4 &= \lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} \\ 4 &= \frac{4}{-3(f(2))^2 f'(2)} \\ 4 &= \frac{4}{-3(1)^2 f'(2)} \\ f'(2) &= \frac{4}{4(-3)} \\ f'(2) &= -\frac{1}{3} \end{aligned}$$

3 ***points:*** 1 ***point for value of*** $f(2)$***,*** 1 ***point for value of*** $f'(2)$***,*** 1 ***point for using L'Hôpital's Rule with justification.***

(d) By definition of continuous, we need to show that $k(2)$ exists, $\lim_{x\to 2} k(x)$ exists, and $\lim_{x\to 2} k(x) = k(2)$. Since we know that $g(x) \le k(x) \le h(x)$ for all $1 < x < 3$, then, in particular, we know that

$$4 = g(2) \le k(2) \le h(2) = 4$$

Hence, $k(2) = 4$.
To compute the limit $\lim_{x\to 2} k(x)$, we need to use the Sandwich Theorem (also known as the Squeeze Theorem):

$$\begin{gathered} g(x) \le k(x) \le h(x) \\ \lim_{x\to 2} g(x) \le \lim_{x\to 2} k(x) \le \lim_{x\to 2} h(x) \\ 4 \le \lim_{x\to 2} k(x) \le 4 \end{gathered}$$

The last inequality is achieved because we know that $g(x)$ and $h(x)$ are continuous, and so we know that

$$\lim_{x\to 2} g(x) = g(2) = 4$$

$$\lim_{x\to 2} h(x) = h(2) = 4$$

So, by the Sandwich Theorem, we know that $\lim_{x\to 2} k(x) = 4$.
Therefore, $\lim_{x\to 2} k(x) = k(2) = 4$, so $k(x)$ is continuous at $x = 2$.
3 ***points:*** 1 ***point for showing*** $k(2) = 4$***,*** 1 ***point for showing*** $\lim_{x\to 2} k(x) = 4$***,*** 1 ***point for justification using Sandwich Theorem.***