AP® CALCULUS BC
2002 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\frac{dy}{dx} = 0$ when $x = 3$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2},$$

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\frac{dy}{dx}$ is negative to the left of $x = 3$ and $\frac{dy}{dx}$ is positive to the right of $x = 3$. Therefore f has a local minimum at $x = 3$.

3 {
- 1 : $x = 3$
- 1 : local minimum
- 1 : justification

(b) $y\,dy = (3-x)\,dx$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

$$8 = 18 - 18 + C\ ;\ C = 8$$

$$y^2 = 6x - x^2 + 16$$

$$y = -\sqrt{6x - x^2 + 16}$$

6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $g(6) = -4$
- 1 : solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

AP® CALCULUS BC
2007 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

(a) $\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2(3x + 2y + 1) = 6x + 4y + 5$	$2: \begin{cases} 1: 3 + 2\frac{dy}{dx} \\ 1: \text{answer} \end{cases}$
(b) If $y = mx + b + e^{rx}$ is a solution, then $m + re^{rx} = 3x + 2\left(mx + b + e^{rx}\right) + 1$. If $r \neq 0$: $m = 2b + 1$, $r = 2$, $0 = 3 + 2m$, so $m = -\frac{3}{2}$, $r = 2$, and $b = -\frac{5}{4}$. OR If $r = 0$: $m = 2b + 3$, $r = 0$, $0 = 3 + 2m$, so $m = -\frac{3}{2}$, $r = 0$, $b = -\frac{9}{4}$.	$3: \begin{cases} 1: \frac{dy}{dx} = m + re^{rx} \\ 1: \text{value for } r \\ 1: \text{values for } m \text{ and } b \end{cases}$
(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0)\cdot\frac{1}{2} = -2 + (-3)\cdot\frac{1}{2} = -\frac{7}{2}$ $f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$ $f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\cdot\frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right)\cdot\frac{1}{2} = -\frac{23}{4}$	$2: \begin{cases} 1: \text{Euler's method with 2 steps} \\ 1: \text{Euler's approximation for } f(1) \end{cases}$
(d) $g'(0) = 3\cdot 0 + 2\cdot k + 1 = 2k + 1$ $g(1) \approx g(0) + g'(0)\cdot 1 = k + (2k + 1) = 3k + 1 = 0$ $k = -\frac{1}{3}$	$2: \begin{cases} 1: g(0) + g'(0)\cdot 1 \\ 1: \text{value of } k \end{cases}$

Question 5

Let f be a function with $f(4) = 1$ such that all points (x, y) on the graph of f satisfy the differential equation

$$\frac{dy}{dx} = 2y(3 - x).$$

Let g be a function with $g(4) = 1$ such that all points (x, y) on the graph of g satisfy the logistic differential equation

$$\frac{dy}{dx} = 2y(3 - y).$$

(a) Find $y = f(x)$.

(b) Given that $g(4) = 1$, find $\lim_{x\to\infty} g(x)$ and $\lim_{x\to\infty} g'(x)$. (It is not necessary to solve for $g(x)$ or to show how you arrived at your answers.)

(c) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection. (It is not necessary to solve for $g(x)$.)

(a) $\frac{dy}{dx} = 2y(3 - x)$

$\frac{1}{y}\,dy = 2(3 - x)\,dx$

$\ln|y| = 6x - x^2 + C$

$0 = 24 - 16 + C$

$C = -8$

$\ln|y| = 6x - x^2 - 8$

$y = e^{6x - x^2 - 8}$ for $-\infty < x < \infty$

5 :
- 1 : separates variables
- 1 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition
- 1 : solution

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) $\lim_{x\to\infty} g(x) = 3$

$\lim_{x\to\infty} g'(x) = 0$

2 :
- 1 : $\lim_{x\to\infty} g(x) = 3$
- 1 : $\lim_{x\to\infty} g'(x) = 0$

(c) $\frac{d^2y}{dx^2} = (6 - 4y)\frac{dy}{dx}$

Because $\frac{dy}{dx} \neq 0$ at any point on the graph of g, the concavity only changes sign at $y = \frac{3}{2}$, half the carrying capacity.

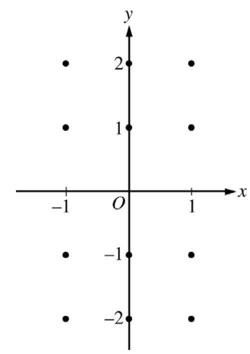
$\left.\frac{dy}{dx}\right|_{y=3/2} = 2\left(\frac{3}{2}\right)\left(3 - \frac{3}{2}\right) = \frac{9}{2}$

2 :
- 1 : $y = \frac{3}{2}$
- 1 : $\left.\frac{dy}{dx}\right|_{y=3/2}$

AP® CALCULUS AB
2010 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.



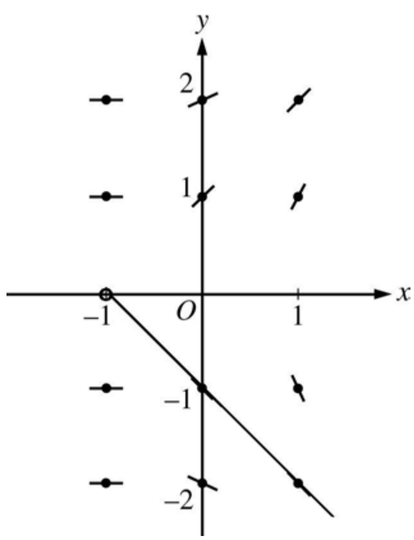
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

(**Note: Use the axes provided in the exam booklet.**)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the *xy*-plane for which $y \neq 0$. Describe all points in the *xy*-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

(a)



$3 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{solution curve through } (0, -1) \end{cases}$

(b) $-1 = \frac{x+1}{y} \Rightarrow y = -x - 1$

$\frac{dy}{dx} = -1$ for all (x, y) with $y = -x - 1$ and $y \neq 0$

1 : description

(c) $\int y\, dy = \int (x+1)\, dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0, -2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

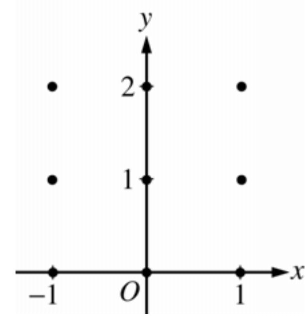
Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

AP® CALCULUS AB
2007 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.



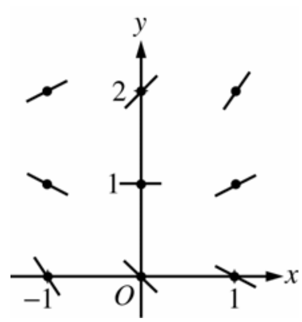
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

(a)



2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

3 : $\begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(0,\,1)} = 0 + 1 - 1 = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,\,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) Substituting $y = mx + b$ into the differential equation:

$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$

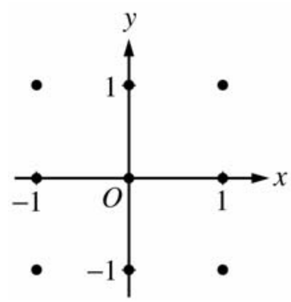
Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

2 : $\begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$

Question 5

Consider the differential equation $\dfrac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

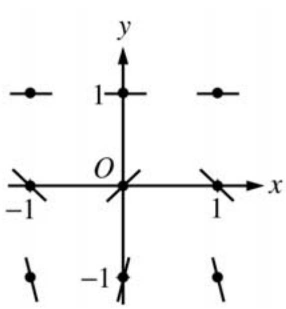
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2: \begin{cases} 1: \text{zero slopes} \\ 1: \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1: c = 1$

(c)

$$\frac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$$

$$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$$

$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$$

$$1 = \frac{1}{\pi}\sin(\pi) + C = C$$

$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$$

$$\frac{\pi}{1-y} = \sin(\pi x) + \pi$$

$$y = 1 - \frac{\pi}{\sin(\pi x) + \pi} \quad \text{for } -\infty < x < \infty$$

$6: \begin{cases} 1: \text{separates variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables