AP® CALCULUS AB
2003 SCORING GUIDELINES (Form B)

Question 6

Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x, where $f(3) = 25$.

(a) Find $f''(3)$.

(b) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = x\sqrt{y}$ with the initial condition $f(3) = 25$.

(a) $$f''(x) = \sqrt{f(x)} + x \cdot \frac{f'(x)}{2\sqrt{f(x)}} = \sqrt{f(x)} + \frac{x^2}{2}$$

$$f''(3) = \sqrt{25} + \frac{9}{2} = \frac{19}{2}$$

$$3 : \begin{cases} 2 : f''(x) \\ \quad < -2 > \text{ product or} \\ \quad \text{chain rule error} \\ 1 : \text{value at } x = 3 \end{cases}$$

(b) $$\frac{1}{\sqrt{y}}\,dy = x\,dx$$

$$2\sqrt{y} = \frac{1}{2}x^2 + C$$

$$2\sqrt{25} = \frac{1}{2}(3)^2 + C\ ;\ C = \frac{11}{2}$$

$$\sqrt{y} = \frac{1}{4}x^2 + \frac{11}{4}$$

$$y = \left(\frac{1}{4}x^2 + \frac{11}{4}\right)^2 = \frac{1}{16}\left(x^2 + 11\right)^2$$

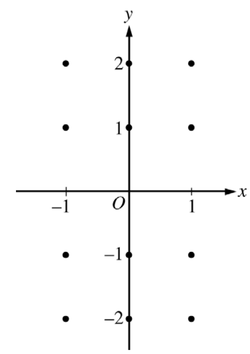
$$6 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(3) = 25 \\ 1 : \text{solves for } y \end{cases}$$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.



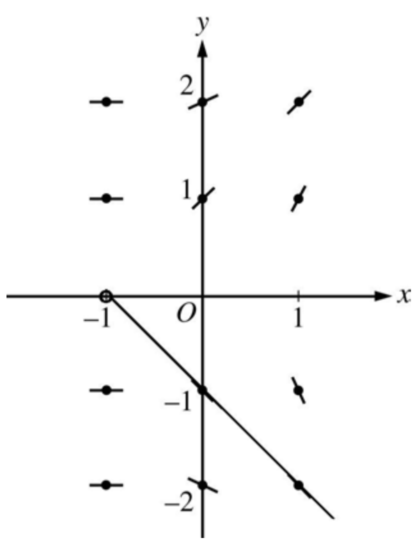
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

(**Note: Use the axes provided in the exam booklet.**)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the *xy*-plane for which $y \neq 0$. Describe all points in the *xy*-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

(a)



$3 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{solution curve through } (0, -1) \end{cases}$

(b) $-1 = \frac{x+1}{y} \Rightarrow y = -x - 1$

$\frac{dy}{dx} = -1$ for all (x, y) with $y = -x - 1$ and $y \neq 0$

1 : description

(c) $\int y\,dy = \int (x+1)\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0, -2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

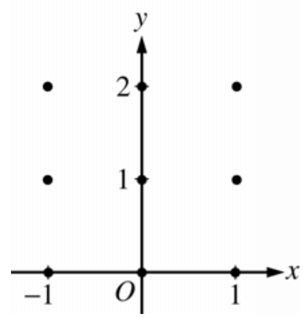
$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.



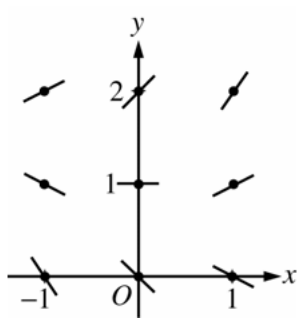
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

(d) Find the values of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

(a)



2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

(b) $\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

$3 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(0,\,1)} = 0 + 1 - 1 = 0$ and $\left.\frac{d^2y}{dx^2}\right|_{(0,\,1)} = 0 + 1 - \frac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) Substituting $y = mx + b$ into the differential equation:

$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$

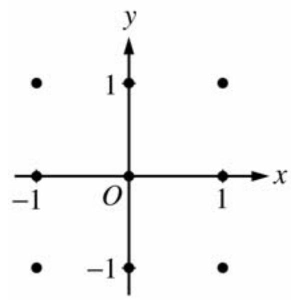
Then $0 = m + \frac{1}{2}$ and $m = b - 1$: $m = -\frac{1}{2}$ and $b = \frac{1}{2}$.

$2 : \begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

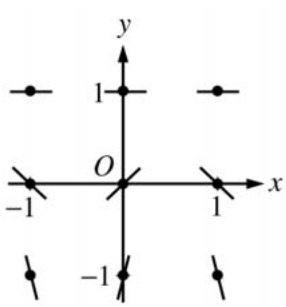
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1 : c = 1$

(c)
$$\frac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$$
$$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$$
$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$$
$$1 = \frac{1}{\pi}\sin(\pi) + C = C$$
$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$$
$$\frac{\pi}{1-y} = \sin(\pi x) + \pi$$
$$y = 1 - \frac{\pi}{\sin(\pi x) + \pi} \quad \text{for } -\infty < x < \infty$$

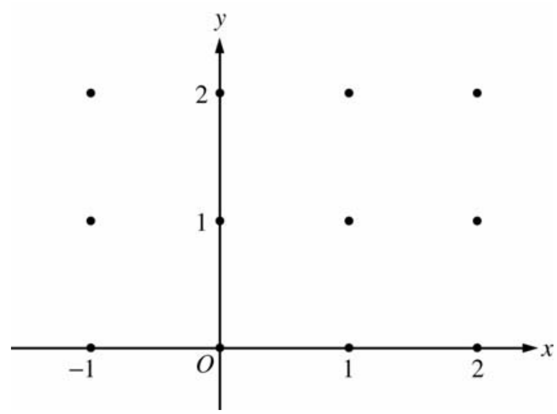
$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 6

Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

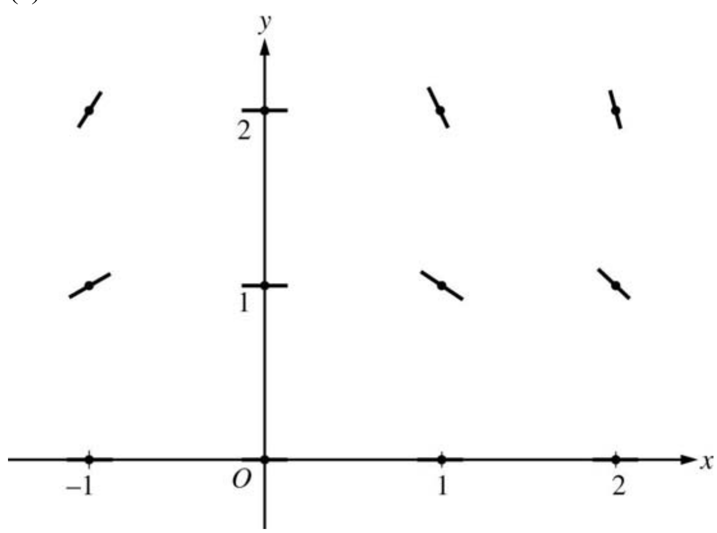


(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) Write an equation for the line tangent to the graph of f at $x = -1$.

(c) Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) Slope $= \frac{-(-1)4}{2} = 2$

$y - 2 = 2(x + 1)$

1 : equation

(c) $\frac{1}{y^2}\,dy = -\frac{x}{2}\,dx$

$-\frac{1}{y} = -\frac{x^2}{4} + C$

$-\frac{1}{2} = -\frac{1}{4} + C;\ C = -\frac{1}{4}$

$y = \frac{1}{\frac{x^2}{4} + \frac{1}{4}} = \frac{4}{x^2 + 1}$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

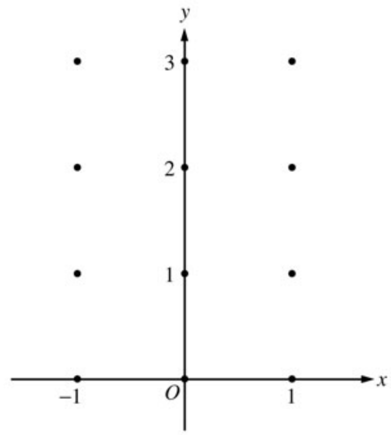
Question 5

Consider the differential equation $\frac{dy}{dx} = x^4(y-2)$.

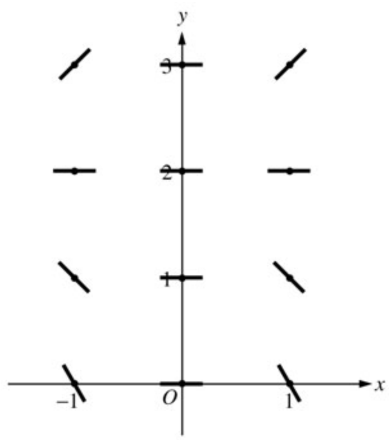
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.



(a)



$2: \begin{cases} 1: \text{zero slope at each point } (x, y) \text{ where } x = 0 \text{ or } y = 2 \\ 1: \begin{cases} \text{positive slope at each point } (x, y) \text{ where } x \neq 0 \text{ and } y > 2 \\ \text{negative slope at each point } (x, y) \text{ where } x \neq 0 \text{ and } y < 2 \end{cases} \end{cases}$

(b) Slopes are negative at points (x, y) where $x \neq 0$ and $y < 2$.

1 : description

(c)
$$\frac{1}{y-2}\,dy = x^4\,dx$$
$$\ln|y-2| = \frac{1}{5}x^5 + C$$
$$|y-2| = e^C e^{\frac{1}{5}x^5}$$
$$y-2 = Ke^{\frac{1}{5}x^5},\; K = \pm e^C$$
$$-2 = Ke^0 = K$$
$$y = 2 - 2e^{\frac{1}{5}x^5}$$

$6: \begin{cases} 1: \text{separates variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \\ \quad 0/1 \text{ if } y \text{ is not exponential} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables