Question 1

A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

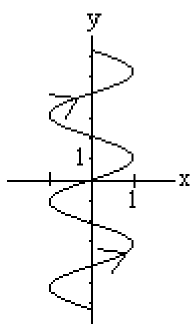
(a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π? Justify your answer.

(a)



$2 \begin{cases} 1 : \text{graph} \\ \quad \text{three cycles of sine} \\ \quad x \text{ between } -1 \text{ and } 1 \\ \quad y \text{ between } -2\pi \text{ and } 2\pi \\ 1 : \text{direction} \end{cases}$

(b) $-1 \le x(t) \le 1$

$-2\pi \le y(t) \le 2\pi$

$2 \begin{cases} 1 : \text{closed interval for } x(t) \\ 1 : \text{closed interval for } y(t) \end{cases}$

(c) $x'(t) = 3\cos 3t = 0$

$3t = \frac{\pi}{2};\ t = \frac{\pi}{6}$

Speed $= \sqrt{9\cos^2(3t) + 4}$

At $t = \frac{\pi}{6}$,

Speed $= \sqrt{9\cos^2\left(\frac{\pi}{2}\right) + 4} = 2$

$3 \begin{cases} 1 : x'(t) = 3\cos 3t = 0 \\ 1 : \text{solves for } t \\ 1 : \text{speed at student's time} \end{cases}$

(d) Distance $= \int_{-\pi}^{\pi} \sqrt{9\cos^2(3t) + 4}\, dt$

$= 17.973 > 5\pi$

$2 \begin{cases} 1 : \text{integral for distance} \\ 1 : \text{conclusion with justification} \end{cases}$

Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi\int_{125}^{275}\left(\frac{B(x)}{2}\right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360}\int_0^{360}\frac{B(x)}{2}\,dx$

2 : { 1 : limits and constant; 1 : integrand }

(b) $\frac{1}{360}\left[120\left(\frac{B(60)}{2}+\frac{B(180)}{2}+\frac{B(300)}{2}\right)\right] =$

$\frac{1}{360}[60(30+30+24)] = 14$

2 : { 1 : $B(60)+B(180)+B(300)$; 1 : answer }

(c) $\frac{B(x)}{2}$ is the radius, so $\pi\left(\frac{B(x)}{2}\right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

2 : { 1 : volume in mm^3; 1 : between $x = 125$ and $x = 275$ }

(d) By the MVT, $B'(c_1) = 0$ for some c_1 in (60, 180) and $B'(c_2) = 0$ for some c_2 in (240, 360). The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2).

3 : { 2 : explains why there are two values of x where $B'(x)$ has the same value; 1 : explains why that means $B''(x) = 0$ for $0 < x < 360$ }

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in (0, 360).

Question 1

A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.

(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.

(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

(d) Find the x-coordinate of the position of the particle at time $t = 3$.

(a) At time $t = 0$:

Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{3^2 + 7^2} = \sqrt{58}$

Acceleration vector $= \langle x''(0), y''(0) \rangle = \langle 0, -3 \rangle$

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration vector} \end{cases}$

(b) $\frac{dy}{dx} = \frac{y'(0)}{x'(0)} = \frac{7}{3}$

Tangent line is $y = \frac{7}{3}(x - 4) + 1$

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \end{cases}$

(c) Distance $= \int_0^3 \sqrt{\left(\sqrt{t^4 + 9}\right)^2 + \left(2e^t + 5e^{-t}\right)^2}\, dt$
$= 45.226$ or 45.227

$3 : \begin{cases} 2 : \text{distance integral} \\ \quad \langle -1 \rangle \text{ each integrand error} \\ \quad \langle -1 \rangle \text{ error in limits} \\ 1 : \text{answer} \end{cases}$

(d) $x(3) = 4 + \int_0^3 \sqrt{t^4 + 9}\, dt$
$= 17.930$ or 17.931

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \sec\left(e^{-t}\right)$$

for $t \geq 0$. At time $t = 1$, the object is at position $(2, -3)$.

(a) Write an equation for the line tangent to the curve at position $(2, -3)$.

(b) Find the acceleration vector and the speed of the object at time $t = 1$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

(d) Is there a time $t \geq 0$ at which the object is on the y-axis? Explain why or why not.

(a) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec\left(e^{-t}\right)}{\tan\left(e^{-t}\right)} = \frac{1}{\sin\left(e^{-t}\right)}$$

$$\left.\frac{dy}{dx}\right|_{(2,-3)} = \frac{1}{\sin\left(e^{-1}\right)} = 2.780 \text{ or } 2.781$$

$$y + 3 = \frac{1}{\sin\left(e^{-1}\right)}(x - 2)$$

$2 : \begin{cases} 1 : \left.\frac{dy}{dx}\right|_{(2,-3)} \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $x''(1) = -0.42253,\ y''(1) = -0.15196$

$a(1) = \langle -0.423, -0.152 \rangle$ or $\langle -0.422, -0.151 \rangle$.

$$\text{speed} = \sqrt{\left(\sec\left(e^{-1}\right)\right)^2 + \left(\tan\left(e^{-1}\right)\right)^2} = 1.138 \text{ or } 1.139$$

$2 : \begin{cases} 1 : \text{acceleration vector} \\ 1 : \text{speed} \end{cases}$

(c) $$\int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.059$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $$x(0) = x(1) - \int_0^1 x'(t)\, dt = 2 - 0.775553 > 0$$

The particle starts to the right of the y-axis.
Since $x'(t) > 0$ for all $t \geq 0$, the object is always moving to the right and thus is never on the y-axis.

$3 : \begin{cases} 1 : x(0) \text{ expression} \\ 1 : x'(t) > 0 \\ 1 : \text{conclusion and reason} \end{cases}$

Question 2

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0,$ the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5,$ find all values of t at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.

(a) The tangent line is vertical when $x'(t) = 0$ and $y'(t) \ne 0$. On $0 < t < 1.5,$ this happens at $t = 1.253$ and $t = 1.144$ or 1.145.	$2: \begin{cases} 1: \text{sets } \frac{dx}{dy} = 0 \\ 1: \text{answer} \end{cases}$
(b) $\left.\frac{dy}{dx}\right\|_{t=1} = \frac{y'(1)}{x'(1)} = 0.863447$ $x(1) = -2 + \int_0^1 x'(t)\,dt = 9.314695$ $y(1) = 3 + \int_0^1 y'(t)\,dt = 4.620537$ The line tangent to the path of the particle at $t = 1$ has equation $y = 4.621 + 0.863(x - 9.315)$.	$4: \begin{cases} 1: \left.\frac{dy}{dx}\right\|_{t=1} \\ 1: x(1) \\ 1: y(1) \\ 1: \text{equation} \end{cases}$
(c) Speed $= \sqrt{(x'(1))^2 + (y'(1))^2} = 4.105$	1 : answer
(d) Acceleration vector: $\langle x''(1), y''(1)\rangle = \langle -28.425, 2.161\rangle$	$2: \begin{cases} 1: x''(1) \\ 1: y''(1) \end{cases}$