Tangential and Normal Components of Acceleration

If a particle is traveling along a curve in space, the coordinates \((x, y, z)\) give its position at time \(t\). However, this is less important to the particle than the direction in which it is heading (the tangent vector), the direction in which it is turning (the normal vector), and the tendency of the path to twist (the binormal vector).

**Definition:** The binormal vector, denoted \(B\), is constructed by taking the cross product of the tangent vector and the normal vector. This ensures that it is of unit length and orthogonal to the plane on which the curve immediately sits at time \(t\). Hence, as the plane of the curve twists, so does the binormal vector

\[ B = T \times N \]

The triple \(T\), \(N\) and \(B\) are called the Frenet frame (after Jean-Frederic Frenet 1816 – 1900) or TNB frame.

The figures above and below illustrate the Frenet frame as it travels up the path of a curve. The blue arrows are the tangent vectors, the green are the normal vectors, and the purple are the binormal vectors. The Frenet frame forms an orthonormal basis for \(\mathbb{R}^3\).
**Acceleration:**
When an object is accelerated by gravity, brakes, or a combination of rocket motors, we usually want to know how much of the acceleration acts in the direction of the motion (the tangential direction) (Thomas, 735). We have previously computed

\[ v = \frac{dr}{dt} = \frac{ds}{dt} = T \frac{ds}{dt} \]

Differentiating both sides of this equality, we achieve

\[
\begin{align*}
a &= \frac{dv}{dt} \\
&= \frac{d}{dt} \left( T \frac{ds}{dt} \right) \\
&= \frac{dT}{dt} \frac{ds}{dt} + T \frac{d^2s}{dt^2} \\
&= \left( \frac{dT}{ds} \frac{ds}{dt} \right) \frac{ds}{dt} + T \frac{d^2s}{dt^2} \\
&= (\kappa N) \left( \frac{ds}{dt} \right)^2 + T \frac{d^2s}{dt^2} \\
&= (\kappa N) |v|^2 + T \frac{d}{dt} |v| \\
&= \frac{ds}{dt} = |v|
\end{align*}
\]

Hence, the acceleration can always be written as a linear combination of the tangent and normal vectors and is independent of the binormal vector. The equation tells us that the acceleration of a particle is independent of the twisting of the path on which it moves. More importantly, the equation tells us exactly how much acceleration occurs in the tangential direction and how much acceleration occurs in the normal direction.
The acceleration always lies in the plane containing $T$ and $N$ and is orthogonal to $B$.

**Lemma 1:** If the acceleration vector is written

$$a = a_T T + a_N N$$

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} \left| v \right|$$

and

$$a_N = \kappa (\frac{ds}{dt})^2 = \kappa \left| v \right|^2$$

are the **tangential** and **normal** scalar components of acceleration.

By definition, acceleration is the rate of change of velocity, $v$. The tangential component of the acceleration, $a_T$, measures the rate of change of the length of the velocity vector because $a_T = \frac{d}{dt} \left| v \right|$. However, the normal component of the acceleration measures the rate of change in the direction of the velocity vector. The normal component of the acceleration vector is the curvature times the square of the speed. This explains why when you are turning sharply, you feel like you have to hold on. If you triple the speed while turning the same radius, the normal component of the acceleration increases by a factor of 9.

Also, if an object moves around a circle at a constant speed, then $\frac{d^2 s}{dt^2} = 0$ and all the acceleration is pointing towards the center of the circle, which is why you always feel like you are being pulled towards the center when you spin around a circle.

To calculate $a_N$, we need to first to understand that since $T$ and $N$ are perpendicular to each other, and their vector sum is the acceleration vector, $a$, then the parallelogram that $T$ and $N$ generate is a rectangle.
By the parallelogram law of addition, \(a\) is the diagonal of the rectangle, and therefore the hypotenuse of a right triangle with legs \(a_T\) and \(a_N\). Therefore, by the Pythagorean theorem,
\[
|a|^2 = a_T^2 + a_N^2
\]
Solving for \(a_N\) yields the formula

**Lemma 2:**
\[
a_N = \sqrt{|a|^2 - a_T^2}
\]

This way we can compute the normal component of the acceleration without computing the curvature first.

**Example:** Without finding \(T\) and \(N\), write the acceleration of the particle traveling along the path
\[
r(t) = (t \cos(t), t \sin(t), t^2)
\]
in the form \(a = a_T T + a_N N\). Then compute the acceleration vector at \(t = 1\).

**Solution:** We use the equations from lemma 1. \(v(t) = (\cos(t) - t \sin(t), \sin(t) + t \cos(t), 2t)\). Now, we must compute \(|v|\),
\[
|v| = \sqrt{(\cos(t) - t \sin(t))^2 + (\sin(t) + t \cos(t))^2 + (2t)^2}
\]
\[
= \sqrt{\cos^2(t) - 2t \cos(t) \sin(t) + t^2 \sin^2(t) + \sin^2(t) + 2t \cos(t) \sin(t) + t^2 \cos^2(t) + 4t^2}
\]
\[
= \sqrt{1 + 5t^2}
\]
The tangential component, if you recall, from lemma 1 is
\[
a_T = \frac{d}{dt} |v|
\]
\[
= \frac{d}{dt} \sqrt{1 + 5t^2}
\]
\[
= \frac{1}{2} (10t)(1 + 5t^2)^{-\frac{1}{2}}
\]
\[
= \frac{5t}{\sqrt{1 + 5t^2}}
\]
Now using that
\[
a = \frac{dv}{dt}
\]
\[
= (- \sin(t) - \sin(t) - t \cos(t), \cos(t) + \cos(t) - t \sin(t), 2)
\]
\[
(-2 \sin(t) - t \cos(t), 2 \cos(t) - t \sin(t), 2)
\]
So by lemma 2, we need only compute the length of the acceleration vector, and we will have the normal component of the acceleration.
\[ a(t) = (-2 \sin(t) - t \cos(t)) + (2 \cos(t) - t \sin(t))^2 + 2^2 \]
\[ = 4 \sin^2(t) + 4t \cos(t) \sin(t) + t^2 \cos^2(t) + 4 \cos^2(t) - 4t \cos(t) \sin(t) + t^2 \sin^2(t) + 4 \]
\[ = 4 + t^2 + 4 \]
\[ = 8 + t^2 \]

So the formula to compute the normal component of acceleration yields

\[ a_N = \sqrt{a(t)^2 - a_T^2} = \sqrt{8 + t^2 - \left(\frac{5t}{\sqrt{1 + 5t^2}}\right)^2} = \sqrt{8 + t^2 - \frac{25t^2}{1 + 5t^2}} \]

At the time \( t = 1 \),

\[ a = \frac{5}{\sqrt{1 + 6}} T + \sqrt{9 - \frac{25}{6}} N \]
\[ = \frac{5}{\sqrt{7}} T + \frac{29}{6} N \]

**Torsion:**

How does the derivative of the binormal vector behave in relation to \( T, N, \) and \( B \). From the rule of differentiating a cross product, we have

\[ \frac{dB}{ds} = \frac{d(T \times N)}{ds} = \frac{dT}{ds} \times N + T \times \frac{dN}{ds} \]

Since \( \frac{dT}{ds} \) is in the direction of the normal vector, then \( \frac{dT}{ds} \times N = 0 \). Therefore,

\[ \frac{dB}{ds} = T \times \frac{dN}{ds} \]

Since \( B \) is a unit vector, we know that it has constant length, and therefore the derivative of the binormal vector is orthogonal to it, i.e.

\[ \frac{dB}{ds} \cdot B = 0 \]

So, therefore, since \( \frac{dB}{ds} \) is orthogonal to the tangent vector \( T \) and the binormal vector \( B \), so it must point in the direction of the normal vector. Hence,

\[ \frac{dB}{ds} = -\tau N \]

The negative sign in the formula above is conventional. The scalar, \( \tau \) (tau), is called the torsion along the curve.

Let \( B = T \times N \). The torsion function of a smooth curve is

\[ \tau = -\frac{dB}{ds} \cdot N \]

Proof:
Unlike the curvature, \( \kappa = \frac{dT}{ds} \), which is never negative, the torsion \( \tau = -\frac{dB}{ds} \cdot N \) can be negative.

The curvature is the rate at which the normal plane turns as the particle moves along the path. The torsion measures how the curve twists.

**FACT:** Torsion may be positive, negative, or zero.

The Frenet frame has three vectors.
- \( T \) is the tangent vector
- \( N \) is the normal vector
- \( B \) is the binormal vector.

The figures above are from the website [http://faculty.eicc.edu/bwood/ma220supplemental/supplemental11.htm](http://faculty.eicc.edu/bwood/ma220supplemental/supplemental11.htm)
FORMULAS FOR COMPUTING CURVATURE AND TORSION

CURVATURE:
\[ \kappa = \frac{\left| \mathbf{\nu} \times \mathbf{a} \right|}{\left| \mathbf{\nu} \right|^3} \]

NOTATION:
\[ \frac{dx}{dt} = \mathbf{x}, \quad \frac{d^2 x}{dt^2} = \mathbf{\xi}, \quad \frac{d^3 x}{dt^3} = \mathbf{\xi} \]

TORSION:
If \( \mathbf{v} \times \mathbf{a} \neq \mathbf{0} \), then
\[ \mathbf{\tau} = \frac{\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{\xi} & \mathbf{\zeta} & \mathbf{\zeta} \\ \mathbf{\xi} & \mathbf{\zeta} & \mathbf{\zeta} \end{vmatrix}}{\left| \mathbf{v} \times \mathbf{a} \right|^2} \]

EXAMPLE 6: Find \( \mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa \), and \( \tau \) for \( \mathbf{r}(t) = (\cos^3 t) \mathbf{i} + (\sin^3 t) \mathbf{j}, 0 < t < \pi/2. \)

SOLUTION: \( \mathbf{v} = (3\cos^2 t)(-\sin t) \mathbf{i} + (3\sin^2 t)(\cos t) \mathbf{j} = (-3\cos^2 t \sin t) \mathbf{i} + (3\sin^2 t \cos t) \mathbf{j} \)
\( \mathbf{a} = ((-6\cos t)(-\sin t)(\sin t) - 3\cos^2 t (\cos t)) \mathbf{i} + (6\sin t \cos t \sin t - 3\sin^2 t \sin t) \mathbf{j} = (6\cos t \sin^2 t - 3\cos^3 t) \mathbf{i} + (6\sin t \cos^2 t - 3\sin^3 t) \mathbf{j} \)
\[ \left| \mathbf{\nu} \right| = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} \]
\[ = \sqrt{9 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} = 3 \sin t \cos t \]
\( \mathbf{T} = \frac{(-3 \cos^2 t \sin t)}{3 \sin t \cos t} \mathbf{i} + \frac{(3 \sin^2 t \cos t)}{3 \sin t \cos t} \mathbf{j} \]
\[ = (-\cos t) \mathbf{i} + (\sin t) \mathbf{j} \]
\[ \frac{d\mathbf{T}}{dt} = (\sin t) \mathbf{i} + (\cos t) \mathbf{j} \]
\[ \left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \]
\( \mathbf{N} = (\sin t) \mathbf{i} + (\cos t) \mathbf{j} \)
\( \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} \]
\[ = 0\mathbf{i} + 0\mathbf{j} - \cos^2 t \mathbf{k} - (\sin^2 t \mathbf{k} + 0\mathbf{i} + 0\mathbf{j}) = -(\cos^2 t + \sin^2 t) \mathbf{k} \]
\[ = -\mathbf{k} \]
As you should notice with this last example, these problems can be long and tedious. The determinants alone are long, and a person can screw up easily when they are doing them. This is where the TI-92 or TI-89 comes in handy. Work through these examples, and use my solutions as a check to make sure that you are doing the work correctly. If you have any questions on any of the examples in this set of notes, please feel free to contact me.

The figures above are from the website http://faculty.eicc.edu/bwood/ma220supplemental-supplemental11.htm