Question 6

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n}$ or $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

$2 \begin{cases} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{cases}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{cases}$

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{cases}$

Question 6

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \ge 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

(a) $f(2) = 1;\ f'(2) = \dfrac{2!}{3};\ f''(2) = \dfrac{3!}{3^2};\ f'''(2) = \dfrac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 : { 1 : coefficients $\dfrac{f^{(n)}(2)}{n!}$ in first four terms; 1 : powers of $(x-2)$ in first four terms; 1 : general term }

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \dfrac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 : { 1 : sets up ratio; 1 : limit; 1 : applies ratio test to conclude radius of convergence is 3 }

(c) $g(2) = 3;\ g'(2) = f(2);\ g''(2) = f'(2);\ g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 : { 1 : first four terms; 1 : general term }

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

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Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	$2: \begin{cases} 1: f(2) = 7 \\ 1: f''(2) = -18 \end{cases}$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	$2: \begin{cases} 1: \text{states } f'(2) = 0 \\ 1: \text{declares } f(2) \text{ as a relative maximum because } f''(2) < 0 \end{cases}$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	$3: \begin{cases} 1: f(0) \approx T(0) = -5 \\ 1: \text{declares that it is not possible to determine} \\ 1: \text{reason} \end{cases}$
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0-2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	$2: \begin{cases} 1: \text{value of Lagrange error bound} \\ 1: \text{explanation} \end{cases}$

Question 2

A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

(a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

1 : answer with reason

(b) $1200 + \int_0^{18} (W(t) - R(t))\, dt = 1309.788$

1310 gallons

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) $W(t) - R(t) = 0$

$t = 0, 6.4948, 12.9748$

t (hours)	gallons of water
0	1200
6.495	525
12.975	1697
18	1310

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or 6.495.

3 : { 1 : interior critical points; 1 : amount of water is least at $t = 6.494$ or 6.495; 1 : analysis for absolute minimum }

(d) $\int_{18}^{k} R(t)\, dt = 1310$

2 : { 1 : limits; 1 : equation }

Question 6

The function f is defined by $f(x) = \frac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\frac{1}{10{,}000}$ of the exact value of the integral?

Solution	Points
(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$	2 : { 1 : first three terms; 1 : general term }
(b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$. $f'(x) = -\left(1+x^3\right)^{-2}\left(3x^2\right)$. Thus, the sum of the series is $f'\left(\frac{1}{2}\right) = -\frac{3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = -\frac{16}{27}$.	2 : { 1 : $f'(x)$; 1 : $f'\left(\frac{1}{2}\right)$ }
(c) $\int_0^x \frac{1}{1+t^3}\,dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{3n+1} + \cdots$	2 : { 1 : first four terms; 1 : general term }
(d) $\int_0^{1/2} \frac{1}{1+t^3}\,dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}$. The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term. $\left\lvert \int_0^{1/2} \frac{1}{1+t^3}\,dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}\right) \right\rvert < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$	3 : { 1 : approximation; 1 : properties of terms; 1 : absolute value of fourth term < 0.0001 }

Question 6

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $$f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$$
$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

3 :
- 1 : two of $6, -2x, \frac{x^2}{3}, -\frac{x^3}{27}$
- 1 : remaining terms
- 1 : general term

$\langle -1 \rangle$ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so
$$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$$
$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

3 :
- 1 : two terms
- 1 : remaining terms
- 1 : general term

$\langle -1 \rangle$ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2k\,e^{-ax/3}$

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$$
$$-2k\,e^{-ax/3} = e^x$$
$$\frac{-a}{3} = 1 \text{ and } -2k = 1$$
$$a = -3 \text{ and } k = -\frac{1}{2}$$

OR

$$f'(x) = -2 + \frac{2}{3}x + \cdots, \text{ so}$$
$$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$$
$$h(x) = 1 + x + \cdots$$
$$-2k = 1 \text{ and } \frac{2}{3}ak = 1$$
$$k = -\frac{1}{2} \text{ and } a = -3$$

3 :
- 1 : computes $k\,f'(ax)$
- 1 : recognizes $h(x) = e^x$, or equates 2 series for $h(x)$
- 1 : values for a and k

Question 6

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

(a) $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$ $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$ $\dfrac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$	3 : { 1 : two of the first four terms; 1 : remaining terms; 1 : general term }
(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.	1 : answer with reason
(c) $\ln\left(1+x^2\right) = \int_0^x \frac{2t}{1+t^2}\,dt$ $= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$ $= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$	2 : { 1 : two of the first four terms; 1 : remaining terms }
(d) $\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$ Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$. Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0, $\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3} \cdot \frac{1}{64} < \frac{1}{100}$.	3 : { 1 : uses $x = \frac{1}{2}$; 1 : value of A; 1 : justification }

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$.
The series converges if and only if $|x+1| < 1$.
Therefore, the interval of convergence is $-2 < x < 0$.

$3: \begin{cases} 1: \text{identifies as geometric} \\ 1: |x+1| < 1 \\ 1: \text{interval of convergence} \end{cases}$

OR

$$\lim_{n \to \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1| < 1 \text{ when } -2 < x < 0$$

At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

OR

$3: \begin{cases} 1: \text{sets up limit of ratio} \\ 1: \text{radius of convergence} \\ 1: \text{interval of convergence} \end{cases}$

(b) Since the series is geometric,

$$f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x} \text{ for } -2 < x < 0.$$

1 : answer

(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln|x| \Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$

$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{value} \end{cases}$

(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$

$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$

$3: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \\ 1: \text{value of } h\left(\frac{1}{2}\right) \end{cases}$

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Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

(a) $$\lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+1}}{(n+1)-1}}{\frac{(2x)^n}{n-1}}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n-1}{n}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n-1}{n}\right| = |2x|$$

$|2x| < 1$ for $|x| < \frac{1}{2}$

Therefore the radius of convergence is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}$.
This is the harmonic series, which diverges.

When $x = \frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$.
This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

5 :
- 1 : sets up ratio
- 1 : limit evaluation
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b)
$$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$
$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$
$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n(2x)^{n-1}\cdot 2}{n-1} + \cdots$$
$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n(2x)^n}{n-1} + \cdots$$
$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n (2x)^n + \cdots$$
$$= 4x^2\left(1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots\right)$$

The series $1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots = \sum_{n=0}^{\infty} (-2x)^n$ is a geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore

$$xy' - y = 4x^2 \cdot \frac{1}{1+2x} \text{ for } |x| < \frac{1}{2}.$$

4 :
- 1 : series for y'
- 1 : series for xy'
- 1 : series for $xy' - y$
- 1 : analysis with geometric series

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Question 6

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$	2 : { 1 : first four terms 1 : general term
(b) The interval of convergence is centered at $x = 0$. At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges. At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges. Therefore the interval of convergence is $-1 < x \le 1$.	2 : answer with analysis
(c) The Maclaurin series for $f'(x)$, $f'\left(t^2\right)$, and $g(x)$ are $f'(x) : \sum_{n=1}^{\infty}(-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$ $f'\left(t^2\right) : \sum_{n=1}^{\infty}(-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$ $g(x) : \sum_{n=1}^{\infty}(-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$ Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.	4 : { 1 : two terms for $f'\left(t^2\right)$ 1 : other terms for $f'\left(t^2\right)$ 1 : first two terms for $g(x)$ 1 : approximation
(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0. Thus $\left\lvert g(1) - \frac{18}{55} \right\rvert < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.	1 : analysis