Remark: Yesterday, we discovered what it meant for a function, \( f \), to be continuous at a point \( x = c \).
We say a function, \( f \), is continuous on an interval \((a, b)\) if it is continuous for all points \( x \in (a, b) \).
(Note: The endpoints are not allowed to be points of continuity because it does not make sense to take the limit at an endpoint. You have to be able to come from both sides.) Sometimes we get lazy and say a function is continuous, if it is continuous on its domain.

1. Properties of Continuous Functions

During our study of limits, we discovered that limits respect addition, multiplication, and composition. Since the definition of continuity at a point requires the limit to exist at that point, then it follows that all the properties of limits also hold for continuity.

**Theorem 1** (Algebra of Continuous Functions). Let \( f \) and \( g \) be continuous functions at \( c \). Then
(a) [Sum] \( f + g \) is continuous at \( c \)
(b) [Difference] \( f - g \) is continuous at \( c \)
(c) [Product] \( fg \) is continuous at \( c \)
(d) [Multiple] \( kf \) is continuous at \( c \), for any real number \( k \). (\( \forall k \in \mathbb{R} \))
(e) [Composition] \( f \circ g \) is continuous at \( c \)

I will give you some practice problems that will require you to apply the above theorem. In justifying your answer you must refer to the theorem. Below is an example

**Example 1.** Find the interval of continuity of the following function.

\[
f(x) = \frac{x \sin(x + 2)}{|x|}
\]

Since \( y = \sin(x) \) and continuous, and \( y = x + 2 \) is continuous, then \( \sin(x + 2) \) is continuous by the Composition Property of Continuous Functions. Furthermore, \( y = x \) is continuous, so \( x \sin(x + 2) \) is continuous by the Product Property of Continuous Functions. Now \( y = \frac{1}{|x|} \) is continuous everywhere but zero, so

\[
f(x) = (x \sin(x + 2)) \left( \frac{1}{|x|} \right) = \frac{x \sin(x + 2)}{|x|}
\]

is continuous on the interval \((-\infty, 0) \cup (0, \infty)\). A faster way of writing this is \( f \) is continuous on \( \mathbb{R} \setminus \{0\} \).

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Problem 1. Find the interval of continuity of the following functions:

1. \( f(x) = x^3 + \sqrt[5]{5x - 3} \)
2. \( f(x) = e^x \sqrt[3]{\sin(2x - 5)} \)
3. \( f(x) = 100x^{14} + 4x^{87} - 34x^{21} + 4351 \)
4. \( f(x) = (\sin(e^{\sqrt[3]{\cos(2x - 5)}}))^{15} \)
5. \( f(x) = \frac{e^{x-3}}{\sqrt[3]{5x}} \)

2. Applications of Continuous Functions

Once you know a function is continuous, you know that it can never jump or break. Suppose you know that \( f(x) \) is continuous, and that at some point \( f(3) = -5 \) and at another \( f(11) = 10 \), then you also know, that between (3,11) it has to hit every value between (-5,11) because it can’t jump. This can be a very powerful tool. Consider the following problem

Example 2. Does there exist a number whose is exactly 2 less than it’s sixth power?

Call this number \( x \), if it exists. Then \( x = x^6 - 2 \) In other words

\[
\begin{align*}
x &= x^6 - 2 \\
0 &= x^6 - x - 2
\end{align*}
\]

Let \( f(x) = x^6 - x - 2 \). Now all we have to do is find a point at which \( f \) is negative and another where \( f \) is positive. Through trial and error, we find \( f(-1) = 1 + 1 - 2 = -2 \) and \( f(2) = 64 - 2 - 2 = 60 \). Therefore, such a number exists.

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